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No. 972

ON TWO-DIMENSIONAL FLOWS OF COMPRESSIBLE FLUIDS

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ON TWO-DIMENSIONAL FLOWS OF COMPRESSIBLE FLUIDS

By Stefan Bergman

SUMMARY

This report is devoted to the study of two-dimensional steady motion of a compressible fluid.

It is shown that the complete flow pattern around a closed obstacle cannot be obtained by the method of Chaplygin. In order to overcome this difficulty, a formula for the stream-function of a two-dimensional subsonic flow is derived. The formula involves an arbitrary function of a complex variable and yields all possible subsonic flow patterns of certain types. It is a generalization of the expression $\text{Im} \left[g(\vec{V}) \right] \rightarrow$ for the stream function of an incompressible fluid. (Here \vec{V} is the velocity vector and g an arbitrary analytic function.)

Conditions are given so that the flow pattern in the physical plane will represent a flow around a closed curve.

The formula obtained can be employed for the approximate determination of a subsonic flow around an obstacle. The method can be extended to partially supersonic flows.

INTRODUCTION

The theory of irrotational two-dimensional flows of an incompressible fluid is based on the theory of analytic functions of a complex variable.

The relation between these two theories is given by the fact that the stream function $\psi(x,y)$ of flow satisfies the Laplace equation $(\partial^2 \psi / \partial x^2) + (\partial^2 \psi / \partial y^2) = 0$. Hence the imaginary part of an analytic function $f(x + iy)$ is a stream

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function of a possible flow, and all flow patterns can be obtained in this way.

For certain purposes, however, it is useful to modify this approach. The stream function may be considered as a function of the components v_1 and v_2 of the velocity vector \vec{v} . Again ψ satisfies the Laplace equation $(\partial^2 \psi / \partial v_1^2) + (\partial^2 \psi / \partial v_2^2) = 0$. Therefore, it is possible to choose as $\psi(v_1, v_2)$ the imaginary part of an analytic function $g(v)$, v being a complex variable in the (v_1, v_2) -plane. In this way the flow pattern in the (v_1, v_2) -plane (hodograph plane) is obtained. In order to find the real shape of the streamlines it is necessary to derive from $\text{Im } g(v)$ the corresponding function of x and y . This transition does not involve any serious theoretical difficulties.

In the case of a potential flow of a compressible fluid the first method (construction of the flow pattern directly in the physical plane) leads to a rather complicated nonlinear partial differential equation. The second approach (construction of the flow pattern in the hodograph plane) reduces to the integration of a linear partial differential equation. (See Chaplygin, reference 1.) Hence, the use of the hodograph method permits the application of various results from the theory of linear partial differential equations. For instance, a stream function in the hodograph plane can be obtained as a linear combination of particular solutions of the linear equation mentioned. Chaplygin was the first to construct a set of such particular solutions. Two other methods of constructing such sets have been given by the present author. (See reference 2, pp. 16-20 and 23-24, and reference 3, sec. 2.)

However, Chaplygin's method and both methods given in references 2 and 3 are not satisfactory in one respect. In general, the stream function will be represented by an infinite series of particular solutions, and such a series will converge only within a part of the domain in which the flow is defined¹.

¹A hodograph of a flow around a profile is (in general) a multiply covered domain (see fig. 1b and 2b) the branch points of which are not necessarily located either at the origin or at infinity; on the other (continued on next page)

To obtain results pertaining to the actual flow, a representation of the stream function as a whole is indispensable. A representation fulfilling these requirements is given in this paper. (See also reference 2, sec. 6, and reference 3, sec. 4.)

If a linear relation between the pressure p and the specific volume $1/\rho$ is assumed:

$$p = A + \sigma/\rho \quad (1)$$

(A, σ constants), then the hodograph equation coincides with the Laplace equation. Assuming relation (1) and using the theory of functions of a complex variable, Von Kármán (reference 4) and Tsien (reference 5) obtained the compressible flow past an elliptic cylinder. Equation (1) is a very rough approximation to the actual pressure-density relation and can be used only in cases where the local velocity is far below that of sound.

In the present report a general pressure-density relation

$$p = A + \sigma \rho^k \quad (2)$$

is used (A, σ, k are constants). (Equation (2) contains as a special case the adiabatic relation $p = \sigma \rho^{1.4}$.) Assuming (2) gives a general formula for the stream function. This formula expresses the stream function of a compressible flow in terms of an arbitrary analytic function of a complex variable.

The representation obtained is, in general, valid in the whole region where the flow is subsonic and in some cases can be extended into a supersonic region also.

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(continued from page 2) hand, the Chaplygin solutions yield flows which (in the hodograph plane) either are single-valued, or multi-valued with a branch point at the origin or at infinity. In order to represent such flow patterns, several series development, each of which represents the stream function ψ under consideration in a certain part of the domain in which ψ is defined, is needed.

I take the opportunity to express my gratitude to Mr. Leonard Greenstone for his assistance in the preparation of the present paper.

NOTATION.

Remark: In dealing with differential equations, the following complex notation is often used:

$$u_z = \frac{\partial u}{\partial z} = \frac{1}{2} \left(\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right), \quad u_{\bar{z}} = \frac{\partial u}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y} \right)$$

$$u_{z\bar{z}} = \frac{1}{4} \Delta u = \frac{1}{4} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \quad z = x + iy, \quad \bar{z} = x - iy$$

$$a = \left[a_0^2 - \frac{1}{2}(k-1)v^2 \right]^{\frac{1}{2}} \quad \text{speed of sound; (equation (28))}$$

a_0 speed of sound at a stagnation point

c (See (94).)

$c(n)$ (See (94).)

$\exp(x) = e^x$; e base of Napierian logarithms

$f(z)$ an arbitrary analytic function of the complex variable z

$$f_{\xi} = \partial f / \partial \xi; \quad f_t = \partial f / \partial t$$

g constant of gravity

$g(\xi)$ an analytic function of the complex variable ξ ; the result of applying the transformation $z = z(\xi)$ to $f(z)$

$$g^{(0)}(\xi) = g(\xi)$$

$$g^{(n)}(\xi) \quad (\text{See (112), ff.}).$$

$h = \sqrt{\frac{k-1}{k+1}}$ for $k > 1$; occasionally the boundary curve of a domain H

k ratio of specific heat at constant pressure to constant volume

$$t(H) = \left(\frac{\partial \Lambda}{\partial H} \right)^2 = \left(\frac{p_0}{\rho(H)} \right)^2 (1 - M^2(H)); \quad (45)$$

p pressure

p_0 pressure "at rest"

r polar coordinate in the physical plane

$$s = 1 - \sqrt{1 - M^2}; \quad (151)$$

schlicht \equiv univalent

v speed; magnitude of \vec{V} ; also, occasionally, the reduced speed v/a_0

v_1, v_2 Cartesian components of \vec{V}

$$w = \phi + i\psi$$

x, y Cartesian coordinates in the physical plane

$$z = x + iy$$

$$\bar{z} = x - iy$$

A constant in the pressure-density relation (22) (See also sec. 3.)

$$D = \frac{\partial(\phi, \psi)}{\partial(x, y)} = \frac{\partial\phi}{\partial x} \frac{\partial\psi}{\partial y} - \frac{\partial\phi}{\partial y} \frac{\partial\psi}{\partial x}; \quad (132)$$

E (See theorem (53).) $E_H = \frac{\partial E}{\partial H}$, $E_\theta = \frac{\partial E}{\partial \theta}$

$$E^* = \exp \left(\int_{-\infty}^{\zeta + \bar{\zeta}} Nd(\zeta + \bar{\zeta}) \right) E; \quad (69)$$

$$F = -(N_{\xi} + N^2) = \frac{(k+1)M^4}{64(1-M^2)^3} \left[-(3k-1)M^4 - 4(3-2k)M^2 + 16 \right]; (71)$$

$$F_m(2\lambda) = \sum_{k=1}^m e^{2\lambda k}; \quad (\text{lemma (67)})$$

F^* (See theorem (83).)

F_{α} (See (115), ff.)

$G(E)$ (See (53); also (124).)

$G_1(E)$ (See (68).)

$G_2(E)$ (See (75).)

$$H = \exp\left(-\int_{-\infty}^{\bar{\xi}+\xi} Nd(\bar{\xi}+\xi)\right) (111)^1; \quad \text{occasionally a domain in the hodograph plane with boundary curve } h$$

H (See (115), ff.)

I heat content

$$I/g \text{ pressure head } \left(\frac{k\sigma\rho^{k-1}}{g(k-1)} \right)$$

Im the imaginary part of

K kernel function (See reference 9; also sec. 3.)

$$K_H \text{ kernel function of } H; \quad K_H^* = \int_0^V K_H(V,A)dV$$

$$L(\psi) \equiv \psi_{\xi}\bar{\eta} + F\psi; \quad (70)$$

$$L_0(\psi) \equiv \psi_{\xi}\bar{\eta} + N(2\lambda) \left[\psi_{\xi} + \psi_{\bar{\eta}} \right]; \quad (46)$$

$$L_m(\psi) \equiv \psi_{\xi}\bar{\eta} + F_m\psi; \quad (74), \text{ ff.}$$

L_{α} (See (115).)

¹See remark 1.

$$M = v/a = v \left[a_0^2 - \frac{1}{2}(k-1)v^2 \right]^{-\frac{1}{2}}; \text{ local Mach number; (31)}$$

$$N = - \frac{(k+1)}{8} \frac{M^4}{(1-M^2)^{3/2}}; \quad (47)$$

$$N_\alpha \quad (\text{See (115).})$$

$$Q^{(1)} = -4 \int_{-\infty}^{\lambda} F \, d\lambda; \quad (107)^1$$

$$Q^{(2)} = \frac{4}{3} F + \frac{1}{6} (Q^{(1)})^2; \quad (108)^1$$

$$Q^{(3)} \quad (\text{See (110).})$$

$$Q^{(n)} \quad (\text{See (84).})$$

$$\tilde{Q}^{(n)} \quad (\text{See (94).})$$

$$R^{(0)} = H \frac{d\lambda}{dv}; \quad (114), \text{ ff.}$$

$$R^{(1)} = \frac{\partial H}{\partial v} + \frac{1}{2} H Q^{(1)} \frac{d\lambda}{dv}; \quad (114), \text{ ff.}$$

$$R^{(n)} \quad (\text{See (114), ff.})$$

Re the real part of

Schlicht \equiv univalent

$$S(\psi) \equiv \left(\frac{p_0}{\rho} \right)^2 (1 - M^2) \psi_{\theta\theta} + \psi_{HH}; \quad (43)$$

$$S_0(\psi) \equiv \left(\frac{p_0}{\rho} \right)^2 (1 - M^2) \psi_{\theta\theta} + \frac{p_0}{\rho} \frac{\partial}{\partial(\log v)} \left[\left(\frac{p_0}{\rho} \right) \frac{\partial \psi}{\partial(\log v)} \right]; \quad (32)$$

$$T = \sqrt{1 - M^2} = \left[\frac{a_0^2 - \frac{1}{2}(k+1)v^2}{a_0^2 + \frac{1}{2}(k-1)v^2} \right]^{\frac{1}{2}}; \quad (51)$$

¹ See Remark 1.

$$\vec{V} = ve^{i\theta} = v_1 + iv_2 \quad \text{velocity vector}$$

$$\vec{V} = ve^{-i\theta} = v_1 - iv_2 = \frac{dw}{dz}$$

V_∞ magnitude of the velocity at infinity

W, W^* (See (119).)

$$X = \frac{s}{2-s} \left\{ 2 \left(\frac{1+h^{-1}-s}{h^{-1}-1+s} \right) \left(\frac{h^{-1}-1}{h^{-1}+1} \right) \right\}^{\frac{1}{h}}; \quad (162)$$

α the angle which a doublet makes with the real axis. (14)
(Also, a real translation of the axis (See (115) ante.)

$\epsilon, a \in B$ "a is a member of B" or "a belongs to B"

$\xi = \lambda \pm i\theta$ in which case $\bar{\xi} = \lambda \mp i\theta$ (This variable use of ξ merely means a reflection with respect to the real axis.)

θ the angle which \vec{V} makes with the real axis

$$\lambda(M) = \Lambda(v); \quad (48), (49)$$

$$\rho \quad \text{density; } \rho = \rho_0 \left[1 - \frac{(k-1)}{2a_0^2} v^2 \right]^{\frac{1}{k-1}}; \quad (25)$$

ρ_0 the density "at rest"

σ a constant in pressure-density relation $p = A + \sigma \rho^k$; (22)

Φ potential function¹; also, the polar angle in the physical plane (polar coordinates)

ψ stream function¹

$$\psi^* = \exp \left(\int_{-\infty}^{\xi+\eta} N d(\bar{\xi}+\eta) \right) \psi; \quad (69)^1$$

Γ circulation; in part II the Gamma function $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$;
(see sec. 11, ff).

¹See remark 1.

Δ Laplace operator: $\Delta\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 4 \left(\frac{\partial^2\phi}{\partial z\partial\bar{z}} \right)$

$\Lambda(H)$ (See (45).)

$\Lambda_H = \partial\Lambda/\partial H, \quad \Lambda_\theta = \partial\Lambda/\partial\theta$

ϕ Potential Function (See remark 1.)

ψ Stream Function (See remark 1.)

Remark 1: In the following, the potential and the stream functions $\phi(x,y)$ and $\psi(x,y)$ (as well as several other variables which are indicated in this section) are considered as functions of different pairs of variables. In passing from the physical to some other plane, new symbols should be introduced for ϕ and ψ , since in different planes ϕ and ψ are different functions of their respective arguments. For instance, passing to the (v,θ) -plane yields

$$\begin{aligned}\phi^{(1)}(v,\theta) &\equiv \phi[x(v,\theta), y(v,\theta)] \\ \psi^{(1)}(v,\theta) &\equiv \psi[x(v,\theta), y(v,\theta)]\end{aligned}$$

For the sake of brevity the author omits the superscript and always writes ϕ and ψ , no matter in which plane he considers these functions.

FOREWORD

The stream function ψ of an incompressible fluid flow is a solution of the Laplace equation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \quad (3)$$

There exists a general formula

$$\psi = \text{Im} \left[f(s) \right], \quad s = \log v - i\theta \quad (4)$$

in terms of an arbitrary function f of one variable, for solutions of this equation. Here v is the speed and θ the angle which the velocity vector forms with the positive x -axis.

In the case of a flow of a compressible fluid, the stream function is a solution of the system of equations¹

$$\frac{\partial^2 \psi}{\partial x^2} \left[1 - \frac{1}{(\rho_0 a_0)^2} \left(\frac{\rho_0}{\rho} \right)^{k+1} \left(\frac{\partial \psi}{\partial y} \right)^2 \right] + \frac{\partial^2 \psi}{\partial y^2} \left[1 - \frac{1}{(\rho_0 a_0)^2} \left(\frac{\rho_0}{\rho} \right)^{k+1} \left(\frac{\partial \psi}{\partial x} \right)^2 \right] \\ + \frac{2}{(\rho_0 a_0)^2} \left(\frac{\rho_0}{\rho} \right)^{k+1} \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} = 0 \quad (5)$$

$$\left(\frac{\rho_0}{\rho} \right)^{1-k} = \left[1 - \frac{k-1}{2(\rho_0 a_0)^2} \left(\frac{\rho_0}{\rho} \right)^2 \left(\left(\frac{\partial \psi}{\partial x} \right)^2 + \left(\frac{\partial \psi}{\partial y} \right)^2 \right) \right]$$

where ρ_0 , a_0 , and k are constant. For air, $k = 1.4$.

A generalization of formula (4) to the case of subsonic flows of a compressible fluid is given in this paper. Let

$M = v / \left[a_0^2 - \frac{1}{2}(k-1)v^2 \right]^{\frac{1}{2}}$ be the local Mach number. If it

is assumed that the flow is subsonic and that $k = 1.4$, functions $\lambda(M)$, $H(M)$, $Q^{(n)}(M)$, $n = 1, 2, \dots$ are determined (see table Ib) so that for the solutions of (5) there is obtained a representation²

¹Stream function ψ and density ρ have to be considered as unknown in system (5). The elimination of ρ , in order to obtain one equation for ψ , is impossible.

²For many purposes in evaluating (6), it suffices to take only a few terms of the series. There also exist methods for improving the convergence of (6). As will be shown elsewhere, it is possible under rather general assumptions to

interchange the $\lim_{m \rightarrow \infty}$ and the summation $\sum_{n=1}^{\infty}$ and thus

obtain a new formula for ψ . In many instances, however, the formula in the original form is more suitable for applications, since by a suitable choice of the m 's it is possible to achieve faster convergence.

$$\psi(M, \theta) = \lim_{m \rightarrow \infty} \operatorname{Im} \left\{ H(M) \left[f(\xi) + \sum_{n=1}^{\infty} Q_m^{(n)}(M) \frac{(2n)!}{2^{2n} n!} \int_0^{\xi} \dots \int_0^{\xi_{n-1}} f(\xi_n) d\xi_n \dots d\xi_1 \right] \right\} \quad (6)$$

$$\xi = \lambda(M) - i\theta, \quad \lambda(M) = \frac{1}{2} \log \left[\frac{1 - (1 - M^2)^{\frac{1}{2}}}{1 + (1 - M^2)^{\frac{1}{2}}} \left(\frac{1 + h(1 - M^2)^{\frac{1}{2}}}{1 - h(1 - M^2)^{\frac{1}{2}}} \right)^{1/h} \right]$$

$$h = \left(\frac{k-1}{k+1} \right)^{\frac{1}{2}}, \quad k > 1$$

in terms of an arbitrary function of one variable¹. Since the transition to the variables x , y does not involve any essential difficulty, (4) and (6) yield patterns for possible incompressible and subsonic (compressible) fluid flows.

Formula (6) is of interest not merely as a tool for computing examples of flows of a compressible fluid, but it may be considered also as an operation which transforms stream functions of incompressible flows into stream functions of compressible flows. The formula suggests the possibility of carrying over various physical laws which govern the motion of an incompressible fluid to the case of a compressible fluid.

In a companion paper this formula will be used for constructing a subsonic flow around a curve which approximates the boundary of an obstacle given in the xy -plane. (See NACA TN No. 973.)

Another application of the above result is to "distortion theory" - that is, the study of how the properties and

¹The possibility for generalization of the formula for the case of a mixed (i.e., partially subsonic and partially supersonic) flow is discussed in the paper. It is observed that for $M < 1$, ξ is a complex number, for $M > 1$ a purely imaginary number. Therefore, for $M < 1$, $f(\xi)$ is a function of one complex variable, while for $M > 1$ it is a function of a real variable.

the shape of the boundary change (in applying the preceding procedure, retain in both formulas (4) and (6), the same function f) upon passing from a flow of an incompressible fluid to the corresponding subsonic flow of a compressible fluid or upon changing the density-pressure relation of the fluid.

I. THE HODOGRAPH METHOD IN THE CASE OF AN INCOMPRESSIBLE FLUID

1. A General Representation for the Stream Function of Flows of an Incompressible Fluid in Terms of an Analytic Function¹ of a Complex Variable

A stream function of a flow of an incompressible perfect fluid is a harmonic function - that is, a function which satisfies the Laplace equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad (7)$$

Conversely, a function ψ which satisfies equation (7) can be interpreted as the stream function of a suitable flow. Since the imaginary part of an analytic function of a complex variable satisfies (7), and for every function satisfying (7) there exists a function $f(z)$ such that

$$\psi(x,y) = \text{Im} [f(z)] \quad (8)$$

(8) is the "general formula" for the stream functions of a flow of an incompressible fluid. Here $f(z)$ ranges over the totality of analytic functions.

In connection with various problems in fluid dynamics as, for example, jet problems, another method of attack was

¹In many instances an analytic function of a complex variable consists of several (or infinitely many) branches, each of which is defined in the whole xy -plane. These branches cover the plane many times. Since a flow covers the plane or a part of it only once, each branch gives rise to a physically possible stream function. However, here and in the following, a function is always spoken of rather than a particular branch of it.

developed the basic idea of which is to consider the flow not in the physical plane but in the hodograph or so-called logarithmic plane¹ - that is, to introduce as independent variables the components v_1, v_2 of the velocity vector

$$\vec{v} = v e^{i\theta} = v_1 + i v_2 = \frac{dw}{dz}, \quad w = \phi + i\psi$$

and $\log v$ and θ , respectively, instead of x and y .

This approach leads to another general formula which, while it is more complicated than (8), has the advantage of being capable of generalization to the case of a compressible fluid.

In the case of an incompressible fluid the stream function

$$\psi(v, \theta) = \psi[x(\log v, \theta), y(\log v, \theta)] \quad (9)$$

is again a harmonic function of $\log v$ and θ and therefore

$$\psi(v, \theta) = \text{Im} [f(\log v - i\theta)] \quad (10)$$

yields a "general formula" for the stream function (considered as a function of $\log v$ and θ). The representation, $\psi(\log v, \theta) = \text{constant}$, for the streamlines (in the logarithmic plane) of the flow is obtained immediately from (10).

2. Passage from the Logarithmic Plane to the Physical Plane

The fact that the flow is considered in the logarithmic plane instead of the original physical plane introduces

¹By the transformation $Z^* = \log Z$ the author passes from the hodograph to the logarithmic plane. In the following, in many instances, it is necessary to pass from the hodograph to the logarithmic plane and vice versa, often without explicitly mentioning it. This fact is stressed here in order to avoid confusion. The plane the Cartesian coordinates of which are $\log v$ and θ is denoted as the logarithmic plane.

See Notation, remark 1.

certain complications. In order to overcome them it is necessary to investigate more thoroughly the relations which exist between the flow around a given obstacle in the physical plane and its image in the hodograph and logarithmic planes.

Suppose that the stream function $\psi = \psi(\log v, \theta)$ of a flow (of an incompressible fluid) in the logarithmic plane is given. The following procedure yields the streamlines of the corresponding flow in the physical plane. Since

$\vec{V} = \frac{dw}{dz}$ (see reference 6, p. 32), it follows inversely that

$$z = \int \frac{dw}{\vec{V}} \quad (11)$$

Writing $\vec{V} = ve^{-i\theta}$ and noticing that the integration occurs along a streamline, $\psi = \text{constant}$, and therefore $d\psi = 0$ gives (11) written in the form

$$\begin{aligned} z &= \int \frac{e^{i\theta}}{v} d\phi = \int \frac{e^{i\theta}}{v} [\phi_v dv + \phi_\theta d\theta] \\ &= \int \frac{e^{i\theta}}{v} \left[\phi_v + \phi_\theta \frac{d\theta}{dv} \right] dv \end{aligned} \quad (12)$$

Using the relations $v\phi_v = -\psi_\theta$ and $\phi_\theta = v\psi_v$ and noticing that along a streamline $d\psi = \psi_v dv + \psi_\theta d\theta = 0$ and therefore, $d\theta/dv = -\psi_v/\psi_\theta$ gives

$$x + iy = z = - \int \frac{e^{i\theta}}{v^2} [\psi_\theta + v^2 \psi_v^2 / \psi_\theta] dv$$

By separating the real and the imaginary parts there is obtained a parametric representation,

$$\begin{aligned} x &= x(v) = - \int_0^v \frac{\cos \theta}{v^2} \frac{[\psi_\theta^2 + v^2 \psi_v^2]}{\psi_\theta} dv \\ y &= y(v) = - \int_0^v \frac{\sin \theta}{v^2} \frac{[\psi_\theta^2 + v^2 \psi_v^2]}{\psi_\theta} dv \end{aligned} \quad (13)$$

for the streamlines in the physical plane. One of the streamlines or a suitable part of it can be used as the boundary of the obstacle.

By employing the hodograph method in airfoil theory it is convenient to use the fact that the approximate form of the hodographs of the flows around airfoils of certain shapes is known. If it is assumed that the domain H which represents the image of the flow in the logarithmic plane is given (see, e.g., fig. 2b), it is possible to construct at first the harmonic function $\psi(\log v, \theta)$ which assumes a constant value on the boundary h of the hodograph and has the prescribed behavior at the point which corresponds to $z = \infty$. Then with (13) the form of the airfoil in the physical plane can be determined.

As is well known, for a given obstacle and a given angle of attack there exists a whole family of flows. If the obstacle has a sharp edge, as occurs in the case of an airfoil, all solutions but one have an infinite velocity at the sharp edge. The Joukowski hypothesis consists of the assumption that this exceptional solution, which has an everywhere finite speed, represents that flow which has physical significance¹.

The hodographs of the flows around the same obstacle (in the physical plane) lead, in general, to quite different pictures in the hodograph, and in the logarithmic planes. For instance, in figures 1a, 1b, and 2a, 2b, two different flows around similar airfoils are indicated. As mentioned before, the hodograph of Joukowski flows has, in general, a shape similar to that indicated in figure 2b. (It is noted that this domain is partially twice covered.)

If the hodograph method is used to obtain the Joukowski flow around some profile, it is at first necessary to determine the function $\psi(\log v, \theta)$ which is defined in the domain H , and has a doublet at A , the point which is the image of $z = \infty$. In order to construct the stream function $\psi(\log v, \theta)$, proceed in the following way: Determine the stream function in the upper half plane (Z -plane) - that is, a function $g(Z)$ which assumes constant values along the real axis and has a combined vortex and doublet at some point, say at $Z = i$, and then the function $Z = Z(\log V)$, which maps the upper plane into the domain H , transforming $Z = i$ into the branch point A of H . As will be seen, a family of solutions is obtained for this problem.

¹A flow fulfilling the conditions of the Joukowski hypothesis is termed a "Joukowski flow."

The axis of the doublet is assumed to form the angle α with the real axis to obtain for the complex potential with the circulation Γ the formula

$$w(Z) = \frac{i\Gamma}{2\pi} \log \frac{Z-i}{Z+i} + \frac{mi}{Z-i} e^{i\alpha} - \frac{mi}{Z+i} e^{-i\alpha} \quad (14)$$

Remark: The term $\frac{i\Gamma}{2\pi} \log \frac{Z-i}{Z+i}$ yields a purely circulatory flow (see fig. 3, also reference 7, p. 326); whereas $mi \left(\frac{e^{i\alpha}}{Z-i} - \frac{e^{-i\alpha}}{Z+i} \right)$ represents a flow without any circulation and with a doublet at $Z = i$ the axis of which intersects the positive Z -axis with the angle α . (See fig. 4, also reference 7, p. 202.)

The question of how to determine the mapping function has a more technical character and will be considered in the next section.

Suppose now a function $\psi = \text{Im} \left[w \left(Z(\bar{V}) \right) \right]$ which assumes a constant value on the boundary h of H . (H is the image of D in the hodograph plane.) The boundary curve of the obstacle is obtained if, starting from some point, say B of h , x and y are determined by integrating along h . However, in general, the obtained curve will not be a closed curve. In order that this be so, it is required that

$$\oint_h \frac{\cos \theta}{v^3} \frac{[\psi_\theta^2 + v^2 \psi_v^2]}{\psi_\theta} dv = 0 \quad (15)$$

$$\oint_h \frac{\sin \theta}{v^3} \frac{[\psi_\theta^2 + v^2 \psi_v^2]}{\psi_\theta} dv = 0 \quad (16)$$

where the integration is carried out along the boundary curve h of H . Thus it is seen that, in order that the obtained boundary in the physical plane be a closed curve, it is necessary to choose Γ , m , and α in such a way that both equations (15), and (16) are satisfied.

Remark: In connection with further applications for a compressible fluid two separate expressions have been derived, one for x , the other for y . Clearly, in the case under consideration they can be combined together, and (15) and (16) are then equivalent to

$$\int_h \frac{dw(\vec{V})}{\vec{V}} = \int_h \frac{dw(Z)}{\vec{V}(Z)} dz \quad (17)$$

Since $w(Z)$ and $\vec{V}(Z)$ are analytic functions of a complex variable Z , which are regular in the upper half plane except at $Z = i$, and $\vec{V}(Z)$ does not vanish there, (17)

equals the residue of $\frac{1}{\vec{V}(Z)} \frac{dw(Z)}{dz}$ at the point $Z = i$.

Write $Z_1 = Z - i$ to obtain for w and \vec{V} , the series developments

$$w = i \frac{\Gamma}{2\pi} \log Z_1 + \frac{mie^{i\alpha}}{Z_1} + \dots, \quad \vec{V} = V_\infty + V_1 Z_1 + \dots \quad (18)$$

$|Z_1| < \rho$, ρ sufficiently small,

and therefore

$$\begin{aligned} \frac{1}{\vec{V}} \frac{dw}{dz_1} &= \frac{-\frac{mie^{i\alpha}}{Z_1^2} + \frac{i\Gamma}{2\pi} \frac{1}{Z_1} + \dots}{V_\infty + V_1 Z_1 + V_2 Z_1^2 + \dots} \\ &= \left[-\frac{mie^{i\alpha}}{Z_1^2 V_\infty} + \frac{i\Gamma}{2\pi} \frac{1}{Z_1} + \dots \right] \left[\frac{1}{V_\infty} - \frac{V_1}{V_\infty^2} Z_1 - \dots \right] \\ &= -\frac{mie^{i\alpha}}{Z_1^2} + \left(\frac{i\Gamma}{2\pi V_\infty} + \frac{V_1 mie^{i\alpha}}{V_\infty^2} \right) \frac{1}{Z_1} + \dots \quad (19) \end{aligned}$$

Thus the above condition becomes

$$\frac{\Gamma}{2\pi} + \frac{V_1 mie^{i\alpha}}{V_\infty} = 0 \quad (20)$$

3. The Determination of the Function $Z = Z(\bar{V})$

Which Maps the Image of the Flow in the
Hodograph Plane into the Upper Half Plane

If the domain H is prescribed, then the function $Z(\bar{V})$ which maps H can be obtained using one of the known methods in the theory of conformal mapping. For instance, Theodorsen's method (see reference 8) may be used to determine the function which maps the circle into H and then compute the inverse function. The theory of orthogonal functions also yields (see reference 9, chs. VI to IX) a simple formula for the function $Z(\bar{V})$.

Denote by $\varphi_v(\bar{V})$ a complete set of orthogonal functions. Such a set can be obtained, for instance, by orthogonalizing the functions $\left\{ \left(\sqrt{\bar{V} - \alpha} \right)^n \right\}$, where α is the branch point of the domain H .

By $K(\bar{V}, \bar{T}) = \sum_{v=1}^{\infty} \varphi_v(\bar{V}) \overline{\varphi_v(\bar{T})}$ is denoted the "kernel function" of the domain. Then the function which maps the domain H into the unit circle, mapping the point A on

the origin, is $\sqrt{\pi} \frac{K_H^*(\bar{V}, \bar{A})}{\sqrt{K_H(A, \bar{A})}}$, and therefore

$$Z(\bar{V}) = -i \frac{\sqrt{\pi} K_H^*(\bar{V}, \bar{A}) - \sqrt{K_H(A, \bar{A})}}{\sqrt{\pi} K_H^*(\bar{V}, \bar{A}) + \sqrt{K_H(A, \bar{A})}} \quad (21)$$

is the required function.

Remark: Equation (10) may be written in a little different form. Writing

$$Z = \log v + i(\pi - \theta)$$

gives

$$\psi(\log v, \theta) = \operatorname{Im} g(Z)$$

where $g(Z) = f(Z - i\pi)$. The passage from θ to $\pi - \theta$ means that in the hodograph plane the domain with respect to the imaginary axis is reflected.

II. THE HODOGRAPH METHOD IN THE CASE OF A COMPRESSIBLE FLUID - SUBSONIC CASE

4. Introduction

In this part the hodograph method will be generalized to the case of a compressible fluid.

The stream function $\psi(x, y)$, in this case, satisfies a complicated nonlinear partial differential equation. (See (26).) If it is assumed that the density pressure relation is of the form

$$p = p(\rho)$$

where $p(\rho)$ is a function of ρ alone, then the use of $\psi(v, \theta)$ instead of $\psi(x, y)$ (as Chaplygin, reference 1, and Molenbroek have shown) represents an important simplification. If the variables v and θ are introduced instead of x and y , the function ψ satisfies a linear partial differential equation $S(\psi) = 0$. (See (30) instead of a nonlinear one, (26).)

Remark: It will be assumed that (unless the contrary is explicitly stated)

$$p(\rho) = A + \sigma \rho^k \quad (22)$$

where A , σ , and k are constants. However, the method developed here can be employed in the case of a much more general pressure density relation.

In the case of an incompressible fluid, instead of merely a statement that the stream function $\psi(\log v, \theta)$ satisfies the Laplace equation, the general formula (10) was given for solutions of the Laplace equation in terms of an arbitrary function f of one variable $\xi = \log v - i\theta$.

The main purpose of the second part of this paper will be to give an analogous formula for a compressible fluid, and to derive from it the representation for the stream function in the physical plane.

As will be proved elsewhere, this result leads to a construction of a flow around an obstacle approximating the given obstacle (in the physical plane).

After a short discussion in section 6 about different types of differential equations and development in section 7 of properties of the auxiliary function $\lambda(v)$ which is needed in the following, an operator is defined in sections 8 to 12 (see (55)), which transforms functions $f(s)$ of one complex variable $s = \lambda(v) + i\theta$ into solutions $\psi(v, \theta)$ of $S_0(\psi) = 0$. Equation $S_0(\psi) = 0$ is the equation for the stream function (in an appropriate plane) for the case of compressible subsonic motion. Then, if the following formulas are used

$$\left. \begin{aligned} x &= x(v) \equiv - \int_0^v \frac{\rho_0 \cos \theta [(1 - M^2)\psi_\theta^2 + v^2 \psi_v^2]}{\rho v^2 \psi_\theta} dv \\ y &= y(v) \equiv - \int_0^v \frac{\rho_0 \sin \theta [(1 - M^2)\psi_\theta^2 + v^2 \psi_v^2]}{\rho v^2 \psi_\theta} dv \end{aligned} \right\} \quad (23)$$

which are derived in section 14 and represent a generalization of (13), a parametric representation for the streamlines in the physical plane of the corresponding flow is obtained.

In section 14 are determined the conditions that the image of the given hodograph yield a flow in the physical plane around a closed curve.

5. Differential Equations for the Potential and Stream Functions

From the continuity and irrotationality of the motion it follows that for every flow there exist two functions, ϕ and ψ , the potential and stream functions, such that

$$\frac{\rho_0}{\rho} \psi_y = \phi_x = v_1, \quad \frac{-\rho_0}{\rho} \psi_x = \phi_y = v_2 \quad (24)$$

Here v_1 and v_2 are the Cartesian components of the velocity vector and ρ is the density. (See reference 6, pp. 228-229 or reference 2, p. 2.) From the Bernoulli relation $1/2 v^2 + I = \text{constant}$, where $I/g = k\rho^{k-1}/g(k-1)$ denotes the pressure head (see reference 6, formulas (13), p. 215, and (10), p. 214), it follows that $1/2 v^2 + \sigma k(k-1)^{-1} \rho^{k-1} = \sigma k(k-1)^{-1} \rho_0^{k-1}$ or

$$\left. \begin{aligned} \rho &= \rho_0 \left[1 - \frac{1}{2} (k-1) k^{-1} \sigma^{-1} \rho_0^{1-k} v^2 \right]^{1/k-1} \\ &= \rho_0 \left[1 - \frac{1}{2} a_0^{-2} (k-1) v^2 \right]^{1/k-1} \\ &= \rho_0 \left[1 - \frac{1}{2} a_0^{-2} (k-1) (\phi_x^2 + \phi_y^2) \right]^{1/k-1} \end{aligned} \right\} \quad (25)$$

If (25) is substituted into (24) there is obtained for ϕ and ψ a system of two nonlinear differential equations

$$\left. \begin{aligned} \psi_y &= \phi_x \left[1 - \frac{1}{2} a_0^{-2} (k-1) (\phi_x^2 + \phi_y^2) \right]^{1/k-1} \\ \psi_x &= -\phi_y \left[1 - \frac{1}{2} a_0^{-2} (k-1) (\phi_x^2 + \phi_y^2) \right]^{1/k-1} \end{aligned} \right\} \quad (26)$$

It is noted that in the case where the motion represents an adiabatic process, $A = 0$, and in the case of air $k = 1.4$.

Eliminating ϕ gives for ψ

$$\left. \begin{aligned} &\frac{\partial^2 \psi}{\partial x^2} \left[1 - \frac{1}{(\rho_0 a_0)^2} \left(\frac{\rho_0}{\rho} \right)^{k+1} \left(\frac{\partial \psi}{\partial y} \right)^2 \right] + \frac{\partial^2 \psi}{\partial y^2} \left[1 - \frac{1}{(\rho_0 a_0)^2} \left(\frac{\rho_0}{\rho} \right)^{k+1} \left(\frac{\partial \psi}{\partial x} \right)^2 \right] \\ &\quad + \frac{2}{(\rho_0 a_0)^2} \left(\frac{\rho_0}{\rho} \right)^{k+1} \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} = 0 \\ &\left(\frac{\rho_0}{\rho} \right)^{1-k} = \left[1 - \frac{2}{2(\rho_0 a_0)^2} \left(\frac{\rho_0}{\rho} \right)^2 \left(\left(\frac{\partial \psi}{\partial x} \right)^2 + \left(\frac{\partial \psi}{\partial y} \right)^2 \right) \right] \end{aligned} \right\} \quad (27)$$

Similarly, eliminating ψ from (26) gives for ϕ :

$$a^2 [\phi_{xx} + \phi_{yy}] = \phi_x^2 \phi_{xx} + 2\phi_x \phi_y \phi_{xy} + \phi_y^2 \phi_{yy} \quad (28)$$

$$a^2 = a_0^2 - \frac{1}{2} (k-1) (\phi_x^2 + \phi_y^2)$$

The derivation of (27) and (28) is omitted here. The equation (28) is derived in reference 6, p. 230. Concerning (27), see reference 10, p. 5.

As Chaplygin and Molenbroek showed, if the variables $\log v$ and θ are introduced instead of x and y , then the equations relating the stream and the potential functions become linear. If¹

$$\left. \begin{aligned} \Phi(v, \theta) &= \Phi[x(v, \theta), y(v, \theta)] \\ \Psi(v, \theta) &= \Psi[x(v, \theta), y(v, \theta)] \end{aligned} \right\} \quad (29)$$

is written, then, instead² of (26), it follows that

$$\left. \begin{aligned} \frac{\rho}{\rho_0} \Phi_{\theta} - \Psi(\log v) &= 0, \quad \left(\frac{\rho_0}{\rho} \right) (1 - M^2) \Psi_{\theta} + \Phi(\log v) = 0 \\ \Psi(\log v) &= \frac{\partial \Psi}{\partial(\log v)} = \frac{v \partial \Psi}{\partial v} \end{aligned} \right\} \quad (30)$$

Here

$$M = v/[a_0^2 - \frac{1}{2}(k-1)v^2]^{1/2} \quad (31)$$

is the local Mach number. Eliminating Φ gives for Ψ a linear equation

$$\begin{aligned} S_0(\Psi) &= \left(\frac{\rho_0}{\rho} \right)^2 (1 - M^2) \Psi_{\theta\theta} \\ &+ \left(\frac{\rho_0}{\rho} \right) \frac{\partial}{\partial(\log v)} \left\{ \left(\frac{\rho_0}{\rho} \right) \frac{\partial \Psi}{\partial(\log v)} \right\} = 0 \end{aligned} \quad (32)$$

In section 11 a general expression³ will be given in the subsonic case (i.e., for $M < 1$) for the solutions of (32) in terms of an arbitrary function of one variable f . That is to say, an expression will be obtained involving an arbitrary function of one variable f such that for every f the obtained expression represents a solution of (32), and conversely every solution of (32) which is regular at the origin can be represented in the afore-mentioned form with a suitably chosen f .

¹See Notation, remark 1.

²A detailed derivation of (30) is given in sec. 3 reference 2.

³It is noted that, in sec. 9, equation (32) is simplified slightly by approximating the coefficients by polynomials. The indicated result refers to this simplified equation.

Remark: It is noted that for $k = -1$, $M < 1$, equation (32) becomes (in appropriate variables) the Laplace equation.

According to (25) and (31),

$$\rho = \rho_0 [1 + (v/a_0)^2]^{-1/2} \quad (33)$$

$$M = (v/a_0) [1 + (v/a_0)^2]^{-1/2} \quad \text{or} \quad (v/a_0) = M[1 - M^2]^{-1/2} \quad (34)$$

$$\text{Since } 1 - M^2 = \frac{1}{1 + (v/a_0)^2}, \quad (\rho_0/\rho)^2 = 1 + (v/a_0)^2$$

and

$$[1 + (v/a_0)^2] \frac{d}{d \log v} = \frac{d}{d \lambda}$$

where

$$\lambda = \frac{1}{2} \log \frac{[1 + (v/a_0)^2]^{1/2} - 1}{[1 + (v/a_0)^2]^{1/2} + 1}$$

equation (32) becomes¹

$$\psi_{\theta\theta} + \psi_{\lambda\lambda} = 0 \quad (35)$$

6. A Remark on Different Types of Equations

The first purpose of the second part of this paper is to give a formula for solutions of (32) in terms of an arbitrary function of one variable.

Before the derivation of this formula is considered, it is well to discuss in some particularly simple cases the "general solutions" of this kind and indicate some characteristic features of such formulas.

The following three equations² will be considered, where μ and δ are real quantities:

¹This result was first obtained by Chaplygin. (See reference 1, p. 99.)

²A differential equation $Au_{MM} + 2Bu_{M\theta} + Cu_{\theta\theta} + Du_{\lambda\lambda} + Eu_{\theta} + F = 0$ is said to be of elliptic or hyperbolic type in the domain R , if $AC - B^2 > 0$ or < 0 in R , respectively.

$$\frac{1}{4} \left(\frac{\partial^2 \psi}{\partial \mu^2} + \frac{\partial^2 \psi}{\partial \delta^2} \right) = \frac{\partial^2 \psi}{\partial \xi \partial \bar{\xi}} = 0, \quad \xi = \mu + i\delta, \quad \bar{\xi} = \mu - i\delta \quad (36)$$

$$\frac{\partial^2 \psi}{\partial \mu \partial \delta} = 0 \quad (37)$$

$$\frac{\partial^2 \psi}{\partial \delta^2} + 4(1 - \mu) \frac{\partial^2 \psi}{\partial \mu^2} - 2 \frac{\partial \psi}{\partial \mu} = 0 \quad (38)$$

In the first case the "general solution" is given by

$$\psi = f(\xi) + g(\bar{\xi}) \equiv f(\mu + i\delta) + g(\mu - i\delta) \quad (39)$$

in the second case by

$$\psi = f(\mu) + g(\delta) \quad (40)$$

where f and g are arbitrary (sufficiently many times differentiable) functions of one variable. As μ and δ are real variables, it is seen that in equation (36) there is an arbitrary function of one complex variable, and in equation (37) two arbitrary functions of a real variable. (Clearly, in equation (36) in order to obtain real solutions, for g must be chosen the conjugate to f ; that is, $\bar{f}(\mu - i\delta)$).

A quite different situation is met in the case of equation (38). By the transformation $\lambda = \sqrt{1 - \mu}$, (38) can be reduced to the form $\frac{\partial^2 \psi}{\partial \delta^2} + \frac{\partial^2 \psi}{\partial \lambda^2} = 0$. The general solution in this case is

$$\psi = f(i\sqrt{1 - \mu} - \delta) + g(i\sqrt{1 - \mu} + \delta) \quad (41)$$

It is seen that the arguments $i\sqrt{1 - \mu} \pm \delta$ are complex for $\mu < 1$ and are real for $\mu \geq 1$.

The solutions behave quite differently than in the previous cases. It may happen that a solution which is real for $\mu < 1$ becomes imaginary for $\mu > 1$. Consider, for instance, the function

$$\frac{1}{i} \left(i(1 - \mu)^{1/2} - \delta \right) + \frac{1}{i} \left(i(1 - \mu)^{1/2} + \delta \right) = (1 - \mu)^{1/2}$$

On the other hand, there also exist solutions which remain real in the whole plane - that is,

On the other hand, there also exist solutions which remain real in the whole plane - for example,

$$(i(1-\mu)^{1/2}-\delta)^2 + (i(1-\mu)^{1/2}+\delta)^2 = 2(\mu-1+\delta^2)$$

Equation (32) is of mixed type¹ and therefore a situation exists of the type exhibited in (38). Of course, the behavior is more complex than in the latter case, because (32) is not the simplest case of equations of this type. First, the function $\lambda(M)$ must be determined; which may be done by reducing the equation (32) to the canonical form.

7. The Function $\lambda(M)$

In this section the function $\lambda(M)$ is introduced. For convenience, an intermediary variable $H = H(v)$ given by

$$\frac{dH(v)}{dv} = \frac{\rho}{v} \quad (42)$$

is employed.

The equation (32) becomes

$$S(\psi) \equiv \left(\frac{\rho_0}{\rho}\right)^2 (1-M^2) \left[\frac{\partial^2 \psi}{\partial \theta^2}\right] + \frac{\partial^2 \psi}{\partial H^2} = l(H) \frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial H^2} = 0 \quad (43)$$

If $M < 1$ - that is, in the case of subsonic motion - then the coefficients of both $\psi_{\theta\theta}$ and ψ_{HH} are positive and therefore the equation is of elliptic type. If $M > 1$ - that is, in the case of supersonic motion - the foregoing coefficients have different signs, and the equation is hyperbolic.

In order to obtain $\lambda(M)$, (43) is reduced to the so-called canonical form. (See reference 11, ch. I, sec. 1.)

Introducing

$$\xi = \Lambda(H) + i\theta, \quad \bar{\xi} = \Lambda(H) - i\theta, \quad (44)$$

where

$$\frac{d\Lambda}{dH} = \rho_0 \rho^{-1} \sqrt{1-M^2} = \sqrt{l(H)}; \quad \text{that is,} \quad \frac{d\Lambda(v)}{dv} = v^{-1}(1-M^2)^{1/2} \quad (45)$$

¹The local Mach number M plays, in the case of (32), a role similar to that of μ in the case of (38).

²It is noted that, for $M < 1$, ξ and $\bar{\xi}$ are complex quantities which are conjugate to each other, for $M > 1$ they become two (independent) purely imaginary quantities. (See (48), (49).)

the equation (43) becomes

$$L_0(\psi) = \frac{\partial^2 \psi}{\partial \xi \partial \bar{\xi}} + \frac{1}{4} \frac{\rho^2}{1 - M^2} \frac{v}{\rho} \left\{ \frac{\partial}{\partial v} \left[\frac{(1 - M^2)^{1/2}}{\rho} \right] \right\} \left[\frac{\partial \psi}{\partial \xi} + \frac{\partial \psi}{\partial \bar{\xi}} \right]$$

$$= \frac{\partial^2 \psi}{\partial \xi \partial \bar{\xi}} + N(\xi + \bar{\xi}) \left[\frac{\partial \psi}{\partial \xi} + \frac{\partial \psi}{\partial \bar{\xi}} \right] \quad (46)$$

where¹

$$N = N(\xi + \bar{\xi}) = N(v)$$

$$= - \frac{(k+1)}{8} v^4 \left[a_0^2 - \frac{1}{2}(k+1)v^2 \right]^{3/2} \left[a_0^2 - \frac{1}{2}(k-1)v^2 \right]^{-1/2}$$

$$= - \frac{(k+1)}{8} \frac{M^4}{(1 - M^2)^{3/2}} \quad (47)$$

The function $\Lambda(v) = \lambda(M)$ may now be easily evaluated. From

$$\frac{d\Lambda(v)}{dv} = \frac{1}{v} \frac{\left[a_0^2 - \frac{1}{2}(k+1)v^2 \right]^{1/2}}{\left[a_0^2 - \frac{1}{2}(k-1)v^2 \right]^{1/2}}$$

it follows, by a purely formal computation, that

$$\Lambda(v) = \lambda(M) = \frac{1}{2} \log \left[\frac{1 - (1 - M^2)^{1/2}}{1 + (1 - M^2)^{1/2}} \left(\frac{1 + h(1 - M^2)^{1/2}}{1 - h(1 - M^2)^{1/2}} \right)^{1/h} \right] \quad (48)$$

$$h = \left(\frac{k-1}{k+1} \right)^{1/2} \quad \text{for } k > 1^2$$

For $M < 1$, $\lambda(M)$ is a real quantity, if $M > 1$, then

$$\lambda(M) = -i \left[(\tan^{-1}) \sqrt{M^2 - 1} - \frac{1}{h} (\tan^{-1}) (h \sqrt{M^2 - 1}) \right] \quad (49)$$

is a purely imaginary quantity.

¹See Notation, remark 1.

$$\text{For } k = 1, \lambda(M) = \frac{1}{2} \log \left[\frac{1 - (1 - M^2)^{1/2}}{1 + (1 - M^2)^{1/2}} \right] + \sqrt{1 - M^2}$$

Remark: If $k < 1$, and $M < 1$,

$$\lambda(M) = \gamma (\tan^{-1}) \left[\gamma \frac{1}{2} (1 - M^2)^{1/2} \right] + \log \left\{ \frac{[1 - (1 - M^2)^{1/2}]}{[1 + (1 - M^2)^{1/2}]} \right\} \quad (50)$$

$$\gamma = \left(\frac{1+k}{1-k} \right)^{1/2}$$

If $k = -1$, $M < 1$

$$\lambda(M) = 1/2 \log \left[\frac{1 - (1 - M^2)^{1/2}}{1 + (1 - M^2)^{1/2}} \right] = 1/2 \log \left\{ \frac{[1 + (v/a_0)^2]^{1/2} - 1}{[1 + (v/a_0)^2]^{1/2} + 1} \right\} \quad (51)$$

or $M = 2e^\lambda / (1 + e^{2\lambda})$ and $(v/a_0) = 2e^\lambda / (1 - e^{2\lambda})$

For the application of this theory, the inverse function $M = M(\lambda)$ often is needed. This can be determined either by preparing once and for all a diagram $\lambda = \lambda(M)$ for a fixed value of k or analytically, representing $M = M(\lambda)$ in the form of an infinite series. For $M < 1$, $k > 1$ there is obtained:

$$\begin{aligned} T(2\lambda) &= \sqrt{1 - M^2(2\lambda)} = 1 - X - 1/2(2k+1)X^2 \\ &- \frac{1}{4}(4k^2 + 6k + 3)X^3 - 1/24(24k^3 + 68k^2 + 76k + 29)X^4 \\ &- \frac{1}{48}(48k^4 + 212k^3 + 392k^2 + 328k + 103)X^5 \\ &- \frac{1}{480}(480k^5 + 2976k^4 + 7968k^3 + 10788k^2 + 7266k + 1935)X^6 \\ &- \frac{1}{2880}(2880k^6 + 23472k^5 + 84232k^4 + 162124k^3 + 173940k^2 \\ &\quad + 98086k + 22675)X^7 - \dots \quad (52) \end{aligned}$$

$$X = 2 \left(\frac{(k+1)^{1/2} - (k-1)^{1/2}}{(k+1)^{1/2} + (k-1)^{1/2}} \right)^{\sqrt{\frac{(k+1)}{(k-1)}}} e^{2\lambda}$$

If M varies between 0 and 1, λ varies between $-\infty$ and 0. The proof of the convergence of (52) for $\lambda < 0$ is given in section 15.

¹The corresponding values of 2λ , M , and v/a_0 for $k = -0.5$ and $k = 1.4$ are given in the tables Ia and Ib.

8. The General Representation for the Solutions of (32)

Theorem (53). Let the function $E(H, \theta, t)$, $-1 \leq t \leq 1$ be a solution of equation

$$G(E) \equiv \left[\frac{\sqrt{1-t^2}}{t(\Lambda + i\theta)} \Lambda_H^2 \left(\frac{E_H}{\Lambda_H} + iE_\theta + \frac{\Lambda_{HH}E}{2\Lambda_H^2} \right) \right]_t + \frac{S(E)}{\sqrt{1-t^2}} = 0^1 \quad (53)$$

which has the property that

$$\left[\left(\frac{E_H}{\Lambda_H} + iE_\theta \right) \frac{\Lambda_H^2 \sqrt{1-t^2}}{t(\Lambda + i\theta)} + \frac{E \sqrt{1-t^2} \Lambda_{HH}}{2t(\Lambda + i\theta)} \right] \quad (54)$$

is continuous at $t = 0$, at $\Lambda = 0$, and at $\theta = 0$. Here S is given by (43), and

$$\Lambda_H \equiv \Lambda_H(H) = \sqrt{l(E)}$$

(See (43) and (45).) Then

$$\Psi(H, \theta) = \int_{-1}^{+1} E(H, \theta, t) f \left[\frac{1}{2}(\Lambda(H) + i\theta)(1-t^2) \right] dt \sqrt{1-t^2} \quad (55)$$

where $f(s)$ is an arbitrary, twice-differentiable function of one variable, will be a solution of $S(\Psi) = 0$.

Proof: It is noticed that

$$f_H = \frac{1}{2} \Lambda_H (1-t^2) f', \quad f_\theta = \frac{i}{2} (1-t^2) f', \quad f_t = -f'(\Lambda + i\theta)t \quad (56)$$

where

$$f'(s) \equiv \frac{df(s)}{ds}$$

Therefore

$$f_H = -\frac{1}{2} \frac{\Lambda_H(1-t^2)}{t(\Lambda + i\theta)} f_t, \quad f_\theta = -\frac{i}{2} \frac{(1-t^2)}{t(\Lambda + i\theta)} f_t, \quad f_\theta = \frac{if_H}{\Lambda_H} \quad (57)$$

Now, by² (55)

¹[]_t indicates differentiation with respect to t .

²In order to be completely rigorous, the integration is carried out along the curve $-1 \leq t \leq -\epsilon$, $t = \epsilon e^{i\varphi}$, $-\pi \leq \varphi \leq 2\pi$, $\epsilon \leq t \leq 1$. Then the integrands in (58) and (59) remain continuous along the path of integration. By use of the property that (54) is continuous, it is possible to let ϵ subsequently approach 0.

$$\left. \begin{aligned} \psi_H &= \int_{-1}^{+1} E_H f \frac{dt}{\sqrt{1-t^2}} + \int_{-1}^{+1} E f_H \frac{dt}{\sqrt{1-t^2}} \\ &= \int_{-1}^{+1} E_H f \frac{dt}{\sqrt{1-t^2}} - \int_{-1}^{+1} \frac{E}{2} \frac{\sqrt{1-t^2} \Lambda_H}{t(\Lambda + i\theta)} \frac{df}{dt} dt \end{aligned} \right\} \quad (58)$$

Integrating by parts gives for the last term in (58):

$$\psi_H = \int_{-1}^{+1} f \left\{ \frac{E_H}{\sqrt{1-t^2}} + \left(\frac{E}{2} \frac{\sqrt{1-t^2} \Lambda_H}{t(\Lambda + i\theta)} \right)_t \right\} dt - \left\{ \frac{E}{2} \frac{\sqrt{1-t^2} \Lambda_H}{t(\Lambda + i\theta)} f \right\}_{t=-1}^1 \quad (59)$$

In an analogous manner there is obtained

$$\psi_\theta = \int_{-1}^{+1} f \left[\frac{E_\theta}{\sqrt{1-t^2}} + \left(\frac{E}{2} \frac{\sqrt{1-t^2} i}{t(\Lambda + i\theta)} \right)_t \right] dt - \frac{E}{2} \frac{\sqrt{1-t^2} i}{t(\Lambda + i\theta)} f \Big|_{t=-1}^1 \quad (60)$$

Now, differentiate (59) with respect to H , (60) with respect to θ , and multiply by Λ_H^2 . There is obtained finally

$$\begin{aligned} \psi_{HH} + \Lambda_H^2 \psi_{\theta\theta} &= \int_{-1}^{+1} f \left\{ \frac{S(E)}{\sqrt{1-t^2}} + \left[\frac{\sqrt{1-t^2}}{t(\Lambda + i\theta)} \Lambda_H^2 \left(\frac{E_H}{2\Lambda_H} + \frac{\Lambda_{HH}}{\Lambda_H^2} \frac{E}{2} + \frac{E_\theta i}{2} \right) \right]_t \right\} dt \\ &+ \int_{-1}^{+1} \left\{ \frac{E_H f_H}{\sqrt{1-t^2}} + \left(\frac{E}{2} \frac{\sqrt{1-t^2}}{t} \right)_t \frac{\Lambda_H f_H}{(\Lambda + i\theta)} + \frac{E_\theta f_\theta \Lambda_H^2}{\sqrt{1-t^2}} \right. \\ &+ \left. \left(\frac{E}{2} \frac{\sqrt{1-t^2}}{t} \right)_t \frac{\Lambda_H^2 f_\theta i}{(\Lambda + i\theta)} \right\} dt \\ &- \left\{ \frac{\sqrt{1-t^2}}{2t(\Lambda + i\theta)} \left[E_H \Lambda_H - \frac{E \Lambda_H^2}{(\Lambda + i\theta)} + E \Lambda_{HH} + E_\theta \Lambda_H^2 i \right. \right. \\ &+ \left. \left. \frac{E \Lambda_H^2}{(\Lambda + i\theta)} \right] \right\} f \Big|_{t=-1}^{t=1} \\ &- \left\{ \frac{\sqrt{1-t^2}}{2t(\Lambda + i\theta)} E \left[\Lambda_H f_H + i \Lambda_H^2 f_\theta \right] \right\} \Big|_{t=-1}^{t=1} \quad (61) \end{aligned}$$

Using the last relation of (57) it is seen that the second and the fourth term in the second integral of (61) cancel each other, and if $f_0 = i f_H / \Lambda_H$ is substituted, the last term of (61) vanishes. Employing (57) again and integrating by parts gives

$$\int_{-1}^{+1} \frac{E_H f_H}{\sqrt{1-t^2}} dt = - \int_{-1}^{+1} \frac{E_H \sqrt{1-t^2} \Lambda_H}{2t(\Lambda + i0)} f_t dt$$

$$= - \left(\frac{E_H \sqrt{1-t^2} \Lambda_H}{2t(\Lambda + i0)} \right) f \Big|_{-1}^{+1} + \int_{-1}^{+1} \left(\frac{E_H \sqrt{1-t^2} \Lambda_H}{2t(\Lambda + i0)} \right)_t f dt \quad (62)$$

and

$$\int_{-1}^{+1} \frac{E_0 f_0 \Lambda_H^2}{\sqrt{1-t^2}} dt = - \int_{-1}^{+1} \frac{E_0 i \Lambda_H^2 \sqrt{1-t^2}}{2t(\Lambda + i0)} f_t dt$$

$$= - \frac{i E_0 \Lambda_H^2 \sqrt{1-t^2}}{2t(\Lambda + i0)} f \Big|_{-1}^{+1} + \int_{-1}^{+1} \left(\frac{i E_0 \Lambda_H^2 \sqrt{1-t^2}}{2t(\Lambda + i0)} \right)_t f dt \quad (63)$$

Using (62) and (63) gives

$$\psi_{HH} + \Lambda_H^2 \psi_{00}$$

$$= \int_{-1}^{+1} f \left\{ \frac{S(E)}{\sqrt{1-t^2}} + \left[\frac{\sqrt{1-t^2}}{t(\Lambda + i0)} \Lambda_H^2 \left(\frac{E_H}{\Lambda_H} + i E_0 + \frac{\Lambda_{HH}}{\Lambda_H^2} \frac{E}{2} \right) \right]_t \right\} dt +$$

$$- \left\{ \frac{\sqrt{1-t^2}}{t(\Lambda + i0)} \left[E_H \Lambda_H + E_0 i \Lambda_H^2 + \frac{1}{2} E \Lambda_{HH} \right] \right\} f \Big|_{t=-1}^{t=1} \quad (64)$$

which implies theorem (53):

9. A Simplification of the Problem

Following the present line of attack the next step is to investigate the solutions of equation (53) and to determine those among them which are most appropriate for the development of the theory.

However, the mathematical analysis of this question has not yet been developed to the extent needed in the case under consideration — that is, in the general case of an equation of mixed type — and to work out this mathematical theory¹

¹The author will develop this approach in a future paper.

here would lead outside the scope of the present paper. Instead of this, two simplifications are made, by which it is possible to employ already known mathematical results.

First, only the subsonic case will be considered. This means that the solutions of the equation (30) will be considered only in the domains where the equation is of elliptic type. Secondly, function F in (70) will be replaced by a polynomial F_m in $e^{2\lambda}$ which vanishes at $\lambda = -\infty$.

In the case of an incompressible fluid where Ψ is a solution of the Laplace equation there is obtained for the stream function the representation

$$\Psi(v, \theta) = \text{Im } f(s), \quad s = \log v - i\theta \quad (65)$$

in terms of an arbitrary function f , of one variable. (See equation (10).)

Generalizing this result, it is found in the following that the stream function of a subsonic flow of a compressible fluid, which is a solution of (32), can be represented in the form

$$\Psi(v, \theta) = \lim_{m \rightarrow \infty} \text{Im} \left\{ H(v) \left[f(\xi) + \sum_{n=1}^{\infty} \frac{(2n)!}{2^n n!} Q_m^{(n)}(v) \int_0^{\xi} \dots \int_0^{\xi_{n-1}} f(\xi_n) d\xi_n \dots d\xi_1 \right] \right\} \quad (66)$$

where $\xi = \lambda(v) + i\theta$, and $H(v)$ and $Q_m^{(n)}(v)$, $n = 1, 2, \dots$ are functions which depend upon m , v , and k . For $k = 1.4$, $Q_m^{(n)} = \lim_{m \rightarrow \infty} Q_m^{(n)}$ are graphically represented in table Ib.

The remainder of this section and sections 10, 11, and 14 are devoted to an exact formulation and derivation of the foregoing representation for the stream function.¹

In order to link the ensuing analysis with standard mathematical procedure, equation (30) is reduced to the canonical form $L_0(\Psi) = 0$, (see (46)), by introducing the variables

¹The mathematical details of the proofs in secs. 9 to 11 may, for the most part, be omitted by the reader whose primary interest lies in the field of physical applications.

$$\xi = \lambda(M) + i0, \quad \bar{\xi} = \lambda(M) - i0 \quad (67)$$

(See (44) and (48).) The equation (53) becomes

$$G_1(E) = (1 - t^2)(E\bar{\xi}_t + NE_t) - t^{-1}(E\bar{\xi} + NE) + 2t L_0(E) = 0 \quad (68)$$

The condition (54) will be satisfied if $(E\bar{\xi} + NE)/t\xi$ is regular at the point $\xi = 0, t = 0$.

If now, instead of ψ and E ,

$$\psi^* = \left[\exp \left(\int_{-\infty}^{\bar{\xi} + \xi} Nd(\bar{\xi} + \xi) \right) \right] \psi \quad \text{and} \quad E^* = \left[\exp \left(\int_{-\infty}^{\bar{\xi} + \xi} Nd(\bar{\xi} + \xi) \right) \right] E \quad (69)$$

are considered, then L_0 becomes

$$L(\psi^*) = \psi^* \xi \bar{\xi} + F \psi^* = 0 \quad (70)$$

where

$$F = -(N\xi + N^2)$$

$$\begin{aligned} &= \left\{ \frac{(k+1)v^4}{\left[a_0^2 - \frac{1}{2}(k+1)v^2 \right]^{3/2} \left[a_0^2 - \frac{1}{2}(k-1)v^2 \right]^{1/2}} \right\} \frac{v}{2(1-M^2)^{1/2}} \\ &= \frac{(k+1)v^8}{64 \left[a_0^2 - \frac{1}{2}(k+1)v^2 \right]^3 \left[a_0^2 - \frac{1}{2}(k-1)v^2 \right]} \\ &= \frac{(k+1)v^4 [16a_0^4 + 4(1-2k)a_0^2 v^2 + (k+1)v^4]}{64 \left[a_0^2 - \frac{1}{2}(k+1)v^2 \right]^3 \left[a_0^2 - \frac{1}{2}(k-1)v^2 \right]} \\ &= \frac{(k+1)M^4 [- (3k-1)M^4 - 4(3-2k)M^2 + 16]}{64(1-M^2)^3} \quad (71) \end{aligned}$$

Since λ and M are connected by the relation (48), the expression F is a function of λ which has a pole of the second order at $\lambda = 0$. And $\lambda = 0$ lies on the boundary of the interval of variation of λ since, if M ranges over $(0, 1)$, λ ranges over $(-\infty, 0)$.

The second simplification is made by replacing the function F by approximating¹ function F which is a polynomial in $e^{2\lambda}$ of the order m and vanishes at $\lambda = -\infty$.

In section 15 it is proved that in every interval $(-\infty, \lambda_0)$, $\lambda_0 < 0$, the original F may be approximated arbitrarily closely by such a polynomial. This means that to every $\lambda_0 < 0$ and every $\epsilon > 0$, there is determined a polynomial F_m in e^λ , $F_m(-\infty) = 0$ such that

$$|F(\lambda) - F_m(e^\lambda)| \leq \epsilon \quad \text{for } -\infty \leq \lambda \leq \lambda_0 \quad (72)$$

The following is now proved.

Lemma (67). To every polynomial $F_m(2\lambda)$ in e^λ there exists a constant c , such that

$$\left| d^K F_m(2\lambda) / d\lambda^K \right| \leq c(K+1)! / (-\lambda)^{K+2}, \quad \text{for } \lambda < 0 \quad \text{and } K = 0, 1, 2, \dots \quad (73)$$

Proof: Since

$$F_m(2\lambda) = \sum_{K=1}^m c_K e^{2\lambda K}, \quad c_K \text{ constants}$$

it suffices to prove that the inequality (73) is valid for the derivatives of a single term $e^{2\lambda K}$. But $d^K e^{2\lambda K} / d\lambda^K = 2^K e^{2\lambda K}$ and as $K \rightarrow \infty$, $K! / (-\lambda)^{K+2} \rightarrow \infty$; therefore there exists a c such that

$$d^K e^{2\lambda K} / d\lambda^K \leq c(K+1)! / (-\lambda)^{K+2}, \quad K = 0, 1, 2, \dots \quad (74)$$

¹In some instances it is expedient to approximate F by the sum of a polynomial and a function which becomes infinite as $\frac{1}{\lambda^2}$ at $\lambda = 0$.

In this second case it is necessary to use the result of reference 3, sec. 4, instead of theorem (83).

Note that in (4.2) of reference 3, $\Gamma(k+1)(\alpha - \lambda)^{k+2}$ should read $\frac{\Gamma(k+1)}{(\alpha - \lambda)^{k+2}}$; in (4.3) $[\Gamma(n+1)]^{-2}$ should be

$[\Gamma(n+1)]^{-1}$; and on line 11 of p. 279 $\lim_{n \rightarrow \infty} c^{(n)} = \frac{1}{2}$ should be $\lim_{n \rightarrow \infty} \left(\frac{c^{(n)}}{c^{(n-1)}} \right)^{1/n} = \frac{1}{2}$.

In sections 10 and 11 an integral representation will be derived in terms of analytic functions for the solutions of L_m , the equation resulting from replacing E by F_m in $L(\psi^*) = 0$; that is, $L_m(\psi^*) = \psi^*_{\xi\xi} + F_m\psi^* = 0$.

10. Lemma

Lemma (75): If $E^*(\xi, \bar{\xi}, t)$ is a solution of equation

$$G_2(E^*) = (1 - t^2)E^*_{\xi\xi} - t^{-1}E^*_{\xi} + 2\xi t[E^*_{\xi\bar{\xi}} + F_mE^*] = 0 \quad (75)$$

and $E^*_{\xi}/\xi t$ is continuous at the point $t = 0$ and $\xi = 0$,
then

$$\int_{-1}^{+1} E^*(\xi, \bar{\xi}, t) f\left(\frac{1}{2}\xi(1 - t^2)\right) dt / (1 - t^2)^{1/2} \quad (76)$$

where $f(z)$, an arbitrary analytic function of a complex variable z , is a solution of

$$\psi^*_{\xi\xi} + F_m\psi^* = 0 \quad (77)$$

Proof: Differentiating with respect to $\bar{\xi}$ gives

$$\psi^*_{\xi\bar{\xi}} = \int_{-1}^{+1} E^*_{\xi} f dt / (1 - t^2)^{1/2} \quad (78)$$

Differentiating again with respect to ξ gives

$$\psi^*_{\xi\xi} = \int_{-1}^{+1} E^*_{\xi\xi} f dt / (1 - t^2)^{1/2} + \int_{-1}^{+1} -E^*_{\xi} f_{\xi} dt / (1 - t^2)^{1/2} \quad (79)$$

If it is noticed that

$$f_{\xi} = \frac{1}{2}(1 - t^2)f' \quad \text{and} \quad f_t = -\xi f'$$

where

$$f' = \frac{df(s)}{ds}, \quad s = \frac{1}{2}\xi(1 - t^2)$$

there is obtained

$$f_{\xi} = -\frac{1}{2} \frac{(1 - t^2)}{\xi t} f_t \quad (80)$$

and therefore

$$\int_{-1}^{+1} \frac{E_{\xi}^* f_{\xi} dt}{(1-t^2)^{1/2}} = -\frac{1}{2} \int_{-1}^{+1} \frac{(1-t^2)^{1/2}}{\xi t} E_{\xi}^* f_{\xi} dt$$

$$= -\frac{(1-t^2)^{1/2}}{2 \xi t} E_{\xi}^* f \Big|_{t=-1}^{t=1} + \int_{-1}^{+1} f \left[\frac{(1-t^2)^{1/2}}{2 \xi t} E_{\xi}^* \right]_t dt \quad (81)$$

(by integration by parts). Substituting the obtained value into (77) gives

$$\psi_{\xi}^* + F_m \psi^* = -\frac{(1-t^2)^{1/2}}{\xi t} E_{\xi}^* f \Big|_{t=-1}^{t=1}$$

$$+ \int_{-1}^{+1} f \left\{ \left[\frac{(1-t^2)^{1/2}}{2 \xi t} E_{\xi}^* \right]_t + \frac{E_{\xi}^*}{(1-t^2)^{1/2}} + F_m E^* \right\} dt \quad (82)$$

which implies lemma (75).

11. The Representation of the Stream Function in the Logarithmic Plane for the Subsonic Case

Theorem (83). Let $F_m(2\lambda)$ be an analytic function of a real variable, λ , defined for $\lambda < 0$, which possesses the property that

$$\left| \frac{d^K F_m}{d\lambda^K} \right| \leq \frac{c(K+1)!}{(-\lambda)^{K+2}} \quad \text{for } \lambda \leq 0 \quad \text{and } K = 0, 1, 2, \dots \quad (83)$$

where c is a suitably chosen constant.

Further, let $Q^{(n)}(2\lambda)$, $n = 1, 2, \dots$, denote a set of functions which are defined by the recurrence formula:

$$\left. \begin{aligned} (2n+1)Q_{\lambda}^{(n+1)} + Q_{\lambda\lambda}^{(n)} + 4F_m Q^{(n)} &= 0, \quad Q_{\lambda}^{(1)} = -4F_m \\ Q^{(n)}(a) &= 0, \quad a < 0 \end{aligned} \right\} \quad (84)$$

Finally, let $g(\xi)$ be an analytic function regular in a domain B which contains the origin. Then

$$\psi^*(\lambda, \theta) = \text{Im} \left[g(\xi) + \sum_{n=1}^{\infty} \frac{(2n)!}{2^{2n} n!} Q^{(n)}(2\lambda) \int_0^{\xi} \dots \int_0^{\xi_{n-1}} g(\xi_n) d\xi_n \dots d\xi_1 \right] \quad (85)$$

will be a solution of

$$\begin{aligned} \frac{1}{4} \Delta \psi^* + F_m(2\lambda) \psi^* &\equiv \frac{1}{4} \left(\frac{\partial^2 \psi^*}{\partial \lambda^2} + \frac{\partial^2 \psi^*}{\partial \theta^2} \right) + F_m(2\lambda) \psi^* \\ &\equiv \psi^*_{\xi\xi} + F_m(2\lambda) \psi^* = 0 \end{aligned} \quad (86)$$

which is defined in every simply connected domain lying in the intersection of H and B, where H denotes the domain $\theta^2 < 3\lambda^2$, $\lambda < 0$.

Proof: If

$$E^* = 1 + t \xi^{1/2} \sum_{n=1}^{\infty} (t \xi^{1/2})^{2n-1} Q^{(n)}(2\lambda) \quad (87)$$

then it must be shown that E^* satisfies the equation

$$G_2(E^*) = (1 - t^2) \frac{E^*}{\xi t} - t^{-1} \frac{E^*}{\xi} + 2 \xi t \left[\frac{E^*}{\xi \xi} + F_m E^* \right] = 0 \quad (88)$$

(see equation 76), and that $E^* / \xi t$ is regular at $\xi = 0$, $t = 0$. E^* formally satisfies equation (88). In fact,

$$\left. \begin{aligned} \frac{E^*}{\xi} &= \frac{1}{2} t^2 \xi Q_{\lambda}^{(1)} + \dots + \frac{1}{2} t^{2n-2} \xi^{n-1} Q_{\lambda}^{(n-1)} + \dots \\ \frac{E^*}{\xi t} &= t \xi Q_{\lambda}^{(1)} + \dots + (n-1) t^{2n-3} \xi^{n-1} Q_{\lambda}^{(n-1)} + \dots \\ \frac{E^*}{\xi \xi} &= t^2 \left(\frac{1}{2} Q_{\lambda}^{(1)} + \frac{1}{4} \xi Q_{\lambda\lambda}^{(1)} \right) + \dots \\ &\quad + t^{2n-2} \left[\frac{1}{2} (n-1) \xi^{n-2} Q_{\lambda}^{(n-1)} + \frac{1}{4} \xi^{n-1} Q_{\lambda\lambda}^{(n-1)} \right] + \dots \end{aligned} \right\} \quad (89)$$

Thus

$$\begin{aligned}
-E_{\xi}^* &= -t \xi Q_{\lambda}^{(1)} - \dots - (n-1) t^{2n-3} \xi^{n-1} Q_{\lambda}^{(n-1)} \\
&\quad - n t^{2n-1} \xi^n Q_{\lambda}^{(n)} - \dots \\
t^2 E_{\xi}^* &= \dots + (n-2) t^{2n-3} \xi^{n-2} Q_{\lambda}^{(n-2)} \\
&\quad + (n-1) t^{2n-1} \xi^{n-1} Q_{\lambda}^{(n-1)} + \dots \\
-\frac{1}{t} E_{\xi}^* &= -\left(\frac{1}{2} t \xi Q_{\lambda}^{(1)} + \dots + \frac{1}{2} t^{2n-3} \xi^{n-1} Q_{\lambda}^{(n-1)} \right. \\
&\quad \left. + \frac{1}{2} t^{2n-1} \xi^n Q_{\lambda}^{(n)} + \dots \right) \\
-2t \xi E_{\xi}^* &= \begin{cases} \dots - (n-2) t^{2n-3} \xi^{n-2} Q_{\lambda}^{(n-1)} \\ \dots - (n-1) t^{2n-1} \xi^{n-1} Q_{\lambda}^{(n)} - \dots \\ \dots - \frac{1}{2} t^{2n-3} \xi^{n-1} Q_{\lambda\lambda}^{(n-2)} \\ \dots - \frac{1}{2} t^{2n-1} \xi^n Q_{\lambda\lambda}^{(n-1)} - \dots \end{cases} \\
-2t \xi F_m E^* &= -2t \xi F_m - \dots - 2t^{2n-3} \xi^{n-1} F_m Q_{\lambda}^{(n-1)} \\
&\quad - 2t^{2n-1} \xi^n F_m Q_{\lambda}^{(n-1)} - \dots
\end{aligned} \tag{90}$$

or

$$G_2(E^*) = -\frac{t \xi}{2} \left[Q_{\lambda}^{(1)} + 4F_m \right] - \dots - \frac{t^{2n-1} \xi^n}{2} \left[(2n-1) Q_{\lambda}^{(n)} \right. \\
\left. + Q_{\lambda\lambda}^{(n-1)} + 4F_m Q_{\lambda}^{(n-1)} \right] - \dots \tag{91}$$

which implies (84). Now, proceed to the proof of the convergence of (87). If A is a dominant of B - that is, if for all derivatives $d^K B / d\lambda^K$, $K = 0, 1, 2, \dots$, it follows that

$$\left| d^K B / d\lambda^K \right| \leq \left| d^K A / d\lambda^K \right|, \quad K = 0, 1, 2, \dots, \quad a \leq \lambda < 0 \tag{92}$$

This will be indicated by

$$B \ll A \quad \text{or} \quad A \gg B \tag{93}$$

By the recurrence formula,

$$(2n+1)\tilde{Q}_\lambda^{(n+1)} = \tilde{Q}_{\lambda\lambda}^{(n)} + 4c(-\lambda)^{-2} \left[\int_a^\lambda \tilde{Q}_\lambda^{(n)} d\lambda + \frac{c^{(n)}}{n} (-\lambda)^{-n} \right] \quad (94)$$

$$\tilde{Q}_\lambda^{(1)} = 4c/(-\lambda)^2, \quad \tilde{Q}^{(n)}(a) = 0$$

the functions $\tilde{Q}^{(n)}$ are introduced.

Writing $\tilde{Q}_\lambda^{(n)} = c^{(n)}(-\lambda)^{-(n+1)}$ gives

$$(2n+1)c^{(n+1)}(-\lambda)^{-(n+2)} = \left((n+1)c^{(n)} + \frac{4cc^{(n)}}{n} \right) (-\lambda)^{-(n+2)} \quad (95)$$

and from this is obtained

$$\lim_{n \rightarrow \infty} \frac{c^{(n+1)}}{c^{(n)}} = \frac{1}{2} \quad (96)$$

Thus, the series

$$1 + \sum_{n=1}^{\infty} \xi^n \tilde{Q}^{(n)}(2\lambda) \quad (97)$$

converges for

$$\left| \frac{\xi}{-2\lambda} \right| < 1 \quad \text{or} \quad \lambda^2 + \theta^2 < 4\lambda^2 \quad (98)$$

It is shown now that

$$Q^{(n)} < \tilde{Q}^{(n)} \quad (99)$$

Clearly, the $\tilde{Q}^{(n)}$ and all derivatives $d^K \tilde{Q}^{(n)}/d\lambda^K$ are positive. Further, by (84), (94), and (83) it follows that

$$Q^{(1)} < \tilde{Q}^{(1)} \quad (100)$$

Equation (99) follows by induction. Suppose it holds for some n , say $n = \mu$; then, by (84) and (94),

$$Q^{(\mu+1)}_{\lambda p} = - \frac{1}{(2\mu+1)} \left[Q^{(\mu)}_{\lambda} + 4 \int_a^{\lambda} E^* Q^{(\mu)} d\lambda \right] \\ \leq \frac{1}{(2\mu+1)} \left[\tilde{Q}^{(\mu)}_{\lambda} + 4c \int_a^{\lambda} \frac{1}{(-\lambda)^2} \tilde{Q}^{(\mu)} d\lambda \right] \leq \tilde{Q}^{(\mu+1)} \quad (101)$$

Further, since all derivatives $\tilde{Q}^{(\mu+1)}_{\lambda p}$ of $\tilde{Q}^{(\mu+1)}$ are combinations of the derivatives $\tilde{Q}^{(\mu)}_{\lambda K}$ of $\tilde{Q}^{(\mu)}$ with positive coefficients, and since the expressions for $Q^{(\mu+1)}_{\lambda p}$ in terms of $Q^{(\mu)}_{\lambda K}$ have the same structure as the expressions for $\tilde{Q}^{(\mu+1)}_{\lambda p}$ in terms of $\tilde{Q}^{(\mu)}_{\lambda K}$, (100) follows. Here $Q^{(\mu)}_{\lambda K} = \frac{\partial Q^{(\mu)}}{\partial \lambda^K}$. Thus the function E^* introduced in (87) is a solution of (88) satisfying the condition: E^*/ξ regular at $\xi = 0$ and $t = 0$.

By lemma (75), it follows that

$$\Psi^* = \text{Im} \left[\int_{-1}^{+1} E^*(\xi, \xi, t) f \left(\frac{1}{2} \xi (1-t^2) \right) dt / (1-t^2)^{1/2} \right] \quad (102)$$

where $f(Z)$, an arbitrary analytic function of a complex variable Z , is a solution of (86). The series (102) converges uniformly for $\theta^2 < 5\lambda^2$. Therefore after replacing E^* in (102) by the right-hand side of (87) the order of summation and integration in the resulting expression may be changed to obtain

$$\Psi^* = \text{Im} \left\{ \int_{-1}^{+1} f \left[\frac{1}{2} \xi (1-t^2) \right] dt / (1-t^2)^{1/2} \right. \\ \left. + \sum_{n=1}^{\infty} Q^{(n)}_{\lambda p} (2\lambda) \xi^n \int_{-1}^{+1} t^n f \left[\frac{1}{2} \xi (1-t^2) \right] dt / (1-t^2)^{1/2} \right\} \quad (103)$$

Let $f(Z) = \sum_{v=0}^{\infty} a_v Z^v$, and write¹

¹ $\Gamma(p)$ is the gamma function $\int_0^{\infty} e^{-t} \times t^{p-1} dt$; so, for integral values of p , $\Gamma(p+1) = p! = p(p-1) \dots 1$.

$$\begin{aligned}
 & \int_{-1}^{+1} f \left[\frac{1}{2} \xi (1-t^2) \right] dt / (1-t^2) = g^{(0)}(\xi) \\
 g^{(n)}(\xi) &= \int_0^{\xi} g^{(n-1)}(\xi_1) d\xi_1 = \sum_{v=0}^{\infty} \frac{a_v \xi^{v+n} \Gamma(v+\frac{1}{2}) \Gamma(\frac{1}{2})}{2^v (n+v) \dots (v+1) \Gamma(v+1)} \\
 & \qquad \qquad \qquad (104)
 \end{aligned}$$

$$n = 0, 1, 2, \dots$$

Then

$$\begin{aligned}
 \xi^n \int_{-1}^{+1} t^{2n} f \left[\frac{1}{2} \xi (1-t^2) \right] dt / (1-t^2)^{1/2} &= \sum_{v=0}^{\infty} a_v \frac{\xi^{n+v}}{2^v} \int_0^{\xi} (1-t)^{v-\frac{1}{2}} t^{n-\frac{1}{2}} dt \\
 &= \sum_{v=0}^{\infty} \frac{1}{2^v} \frac{\Gamma(v+\frac{1}{2}) \Gamma(n+\frac{1}{2})}{\Gamma(n+v+1)} \xi^{n+v} a_v \\
 &= \left(n - \frac{1}{2} \right) \left(n - \frac{3}{2} \right) \dots \frac{1}{2} \sum_{v=0}^{\infty} \frac{a_v \Gamma(v+\frac{1}{2}) \Gamma(\frac{1}{2})}{(n+v) \dots (v+1) \Gamma(v+1)} \frac{\xi^{n+v}}{2^v} \\
 &= \frac{(2n-1)(2n-3) \dots 1}{2^n} g^{(n)}(\xi) = \frac{\Gamma(2n+1)}{2^{2n} \Gamma(n+1)} g^{(n)}(\xi) \quad (105)
 \end{aligned}$$

Substitute the last term of (105) into (103) to obtain the expression (85).

12. The Evaluation of the Coefficients $Q^{(n)}(M)^1$

It was proved in section 9 that if F is replaced by F_m the series obtained for E^* converges, and (85) represents a solution of (86).

It is important for practical purposes to compute the $Q^{(n)}$ explicitly.² Since m can be chosen so large that in

¹In this and the following section advantage is taken of the remark in footnote 2 on p. 10.

²In a later paper the corrections to be made in order to pass from the $Q^{(n)}$ obtained in the above-described manner to the $Q_m^{(n)}$ obtained using a polynomial F_m will be determined. It will be seen that, in general, these corrections may be neglected.

the given interval $(-\infty, \lambda_0)$, $\lambda_0 < 0$, F_m , and any required number of the derivatives of F_m differs by less than any prescribed $\epsilon > 0$ from the corresponding derivatives of F , the $Q^{(n)}$ will be computed using the function F instead of F_m . As will be shown, the expressions obtained for $Q^{(n)}$ consist of a finite number of rational and logarithmic terms in

$$T = (1-M^2)^{1/2} \quad (106)$$

(For the relation between λ and M , see (48).) From the second equation of (84)

$$\begin{aligned} Q^{(1)} &= -4 \int_{-\infty}^{\lambda} F d\lambda = -2 \int_1^{T(2\lambda)} F \frac{d(2\lambda)}{dT} dT \\ &= -\frac{1}{8} \int_1^T \left[\frac{5(1+k)}{T^6} - \frac{12k}{T^4} + \frac{2(3k-7)}{T^2} + 4(k+2) \right. \\ &\quad \left. - (3k-1)T^2 \right] \frac{T^2}{(T^2-1)(1-h^2T^2)} dT \\ &= \frac{(k+1)}{8} \left[-\frac{5}{3T^3} + \frac{2k}{(k+1)T} + \frac{1-3k}{k-1} T \right. \\ &\quad \left. + \frac{4}{(k^2-1)k^2-1} \log \frac{1+\frac{k-1}{k+1}T}{1-\frac{k-1}{k+1}T} \right] \Bigg|_{T=1}^T \quad (107) \end{aligned}$$

Setting $n = 1$ in the first expression of (84) gives

$$\begin{aligned} 3Q_{\lambda}^{(2)} &= -Q_{\lambda\lambda}^{(1)} - 4FQ_{\lambda}^{(1)} \\ \text{or} \\ Q^{(2)} &= -\frac{1}{3} Q_{\lambda}^{(1)} \Bigg|_{\lambda=-\infty}^{\lambda} - 4 \int_{-\infty}^{\lambda} F Q_{\lambda}^{(1)} d\lambda = -\frac{1}{3} Q_{\lambda}^{(1)} + \int Q^{(1)} dQ^{(1)} \\ &= -\frac{1}{3} Q_{\lambda}^{(1)} + \frac{1}{6} Q^{(1)2} = \frac{4}{3} F + \frac{1}{6} Q^{(1)2} \quad (108) \end{aligned}$$

It is noted that, in general, it follows from (84) that

$$(2n+1)Q^{(n+1)} = -Q_{\lambda}^{(n)} - 4 \int_{-\infty}^{\lambda} Q^{(n)} F d\lambda = -Q_{\lambda}^{(n)} + \int_{T(-\infty)}^{T(\lambda)} Q^{(n)} dQ^{(1)} \quad (109)$$

since from $(d^k F/d\lambda^k)_{\lambda=-\infty} = 0$ it follows that

$$Q_{\lambda}^{(n)}(-\infty) = Q_{\lambda\lambda}^{(n)}(-\infty) = 0, \quad n = 0, 1, 2, \dots$$

Thus there is obtained

$$5Q^{(3)} = -\frac{4}{3}F_{\lambda} + \frac{4}{3}FQ^{(1)} - \frac{16}{3} \int_{-\infty}^{\lambda} F^2 d\lambda + \frac{1}{18} Q^{(1)^3}$$

or

$$Q^{(3)} = -\frac{4}{15}F_{\lambda} + \frac{4}{15}FQ^{(1)} - \frac{16}{15} \int_{-\infty}^{\lambda} F^2 d\lambda + \frac{1}{90} Q^{(1)^3} \quad (110)$$

where

$$F_{\lambda} = \frac{(k+1)^2}{128} \frac{(T^2-1)}{T^9} \left(1 - \frac{k-1}{k+1} T^2\right) \left[\begin{aligned} &-30(k+1) + 48kT^2 \\ &-4(3k-7)T^4 - 2(3k-1)T^6 \end{aligned} \right]$$

and

$$\begin{aligned} \int_{-\infty}^{\lambda} F^2 d\lambda &= \frac{25(k+1)^2}{9T^9} - \frac{10k(k+1)}{T^7} + \frac{39k^2-80k-115}{5T^5} \\ &+ \frac{4(11k^2+48k^2+59k+20)}{3(k+1)T^3} - \frac{41k^4-96k^3-18k^2+112k+91}{(k+1)^2 T} \\ &+ \frac{2(3k^4+17k^3-15k^2-21k+8)}{(k-1)^2} T - \frac{(3k-1)^2(k+1)}{3(k-1)} T^3 \end{aligned}$$

In order to obtain ψ from ψ^* (see (69)) it is necessary to have

$$H = \exp \left(- \int_{-\infty}^{\xi+\xi} N(d\xi+\xi) \right) = \frac{1}{(1-M^2)^{1/4}} \left[\frac{2}{2+(k-1)M^2} \right]^{\frac{1}{2(k-1)}} \quad (111)$$

Thus

$$\psi(v, \theta) = H \left[\text{Im } g(\xi) + \sum_{n=1}^{\infty} \frac{\Gamma(2n+1)}{2^{2n} \Gamma(n+1)} Q^{(n)}(\xi) \text{Im } g^{(n)}(\xi) \right] \quad (112)$$

where Im = imaginary part and $g^{(n+1)}(\xi) = \int_0^{\xi} g^{(n)}(\xi) d\xi$, $g^{(0)}(\xi) = g(\xi)$. In order to evaluate the terms of (136) ψ_{θ} and ψ_v are needed.

Differentiating (112) with respect to θ and to v , respectively, gives

$$\psi_{\theta} = H \left[\text{Re } g_{\xi} + \frac{1}{2} Q^{(1)} \text{Re } g + \frac{3}{4} Q^{(2)} \text{Re } g^{(1)} + \dots \right] \quad (113)$$

(Re = real part)

$$\begin{aligned} \psi_v = & R^{(0)} \text{Im } g_{\xi} + R^{(1)} \text{Im } g + \frac{1}{2} R^{(2)} \text{Im } g^{(1)} + \frac{3}{4} R^{(3)} \text{Im } g^{(2)} \\ & + \dots + \frac{\Gamma(2n-1)}{2^{2n-2} \Gamma(n)} R^{(n)} \text{Im } g^{(n-1)} + \dots \end{aligned} \quad (114)$$

where $R^{(0)} = H \frac{d\lambda}{dv}$, $R^{(1)} = H_v + \frac{1}{2} H Q^{(1)} \frac{d\lambda}{dv}$

$$R^{(n)} = \left[\left(H Q^{(n-1)} \right)_{\lambda} + \frac{2n(2n-1)}{4n} H Q^{(n)} \right] \frac{d\lambda}{dv} \quad n = 2, 3, \dots$$

The values of H , $Q^{(n)} = \lim_{m \rightarrow \infty} Q_m^{(n)}$, for $k = -0.5$ and those of H , $Q^{(n)}$, $R^{(n)} = \lim_{m \rightarrow \infty} R_m^{(n)}$ for $k = 1.4$ are given numerically in tables IIa and IIb, respectively.¹ Tables Ia and Ib give their graphical representations.

Remark: If the origin is moved and λ is replaced by $\lambda^* + \alpha$; that is, setting

$$\xi^* = \xi + \alpha, \quad \bar{\xi}^* = \bar{\xi} + \alpha, \quad \alpha \text{ real,}$$

equation (70) assumes the form

¹Mr. E. Ostrow assisted with the computation of these tables.

$$L_{\alpha}(\psi) \equiv \frac{\partial^2 \psi}{\partial \xi^* \partial \bar{\xi}^*} + N_{\alpha} \left(\frac{\partial \psi}{\partial \xi^*} + \frac{\partial \psi}{\partial \bar{\xi}^*} \right) = 0 \quad (115)$$

The general formula (see (85) and (77)) may now be used for the solutions of (115).

Since $F = -(N_{\xi} + N_{\bar{\xi}})$ (see (71)) it is found that $F_{\alpha} = F(\xi^* - \alpha, \bar{\xi}^* - \alpha)$. Using (109) gives

$$Q_{\alpha}^{(n)} = Q^{(n)}(2\lambda^* - 2\alpha), \quad H_{\alpha} = H(2\lambda^* - 2\alpha).$$

Thus, the generalized formula (112) becomes

$$\begin{aligned} \Psi(v, \theta) = & H[2\lambda^*(v) - 2\alpha] \operatorname{Im} \left[g(\xi^*) \right. \\ & \left. + \sum_{n=1}^{\infty} \frac{(2n)!}{2^{2n}(n)!} Q^{(n)}(2\lambda^*(v) - 2\alpha) g^{(n)}(\xi^*) \right] \quad (116) \end{aligned}$$

$$g^{(n+1)}(\xi^*) = \int_0^{\xi^*} g^{(n)}(\xi^*) d\xi^*, \quad \xi^* = \lambda^*(v) + i\theta$$

13. The Behavior of a Subsonic Flow at Infinity

At the point $a = \alpha + i\beta$, α, β real, of the hodograph plane which corresponds to the point¹ $z = \infty$ of the physical plane, the stream function Ψ has a singularity. This fact leads to the study of the singularities of functions satisfying (46).

If point a is a branch point, then the use of formulas (85) and (69) yields a singularity which possesses the desired features. Indeed,

$$\begin{aligned} \Psi(\lambda(v), \theta) = & H(v) \left\{ \operatorname{Im} \left[\frac{1}{(\alpha - \xi)^{1/2}} - Q^{(1)} \left((\alpha - \xi)^{1/2} - \alpha^{1/2} \right) \right. \right. \\ & \left. \left. + \frac{3}{2} Q^{(2)} \left((\alpha - \xi)^{3/2} - \alpha^{1/2} \xi - \frac{2}{3} \alpha^{3/2} \right) + \dots \right] \right\} \\ & \xi = \lambda(v) - i\theta \quad (117) \end{aligned}$$

¹This means that the velocity $ae^{-i\theta}$ obtains at point $z = \infty$ (of the physical plane).

is a stream function which is two-valued at point a and becomes infinite as $1/(a - \xi)^{1/2}$ - that is, it behaves like $1/(a - \log v + i\theta)^{1/2}$ for the case of an incompressible fluid.

If, however, point a is not a branch point - that is, if (85) and (69) are applied to the function $g = 1/(a - \xi)$ - then

$$\psi\left(\lambda\left(\frac{v}{a_0}\right), \theta\right) = H(v) \operatorname{Im} \left\{ \frac{1}{a - \lambda(v)} + \frac{1}{2} Q^{(1)}(v) (\log(a - \xi) - \log a) + \frac{4}{3} Q^{(2)}(v) ((a - \xi) \log(a - \xi) - (a - \xi) \log(a + \xi)) + \dots \right\} \quad (118)$$

is obtained, which is not a single-valued function.

For the sake of brevity the case of $g = \log(a - \xi)$ is not discussed, but here also, in general, a many-valued function is obtained.

The function (118) can, however, be made one-valued by replacing the many-valued term ($\theta = \arg \xi$) by its mean value (in the sheet under consideration).

Clearly, this new function will no longer be an exact solution of equation (46), but in many instances it will not differ very much from an exact solution.¹ Plainly this procedure may be refined.

It is, however, of interest from a theoretical point of view to determine (exact) solutions of² (86) which are single-valued and have a logarithmic singularity at point a .

Clearly, it is sufficient to find functions for equation (86) which possess a logarithmic singularity.

A function

$$\begin{aligned} W^*(\lambda, \theta; \lambda_0, \theta_0) &= W(\xi, \bar{\xi}; \xi_0, \bar{\xi}_0) \\ &= A(\xi, \bar{\xi}; \xi_0, \bar{\xi}_0) \log |\xi - \xi_0| + B(\xi, \bar{\xi}; \xi_0, \bar{\xi}_0) \end{aligned} \quad (119)$$

¹It is noted that in this case expressions (136) will no longer be complete differentials, which fact may cause some difficulty if a is an interior point of the domain.

²It is assumed here that F is replaced by F_m . (See sec. 9.)

which, considered as a function of $\xi, \bar{\xi}$, satisfies equation (86) in the whole plane, except at the point $\xi = \xi_0$, is termed a "fundamental solution" of equation (86) with the affix at $\xi = \xi_0$.

Clearly, $\psi = W(\xi, \bar{\xi}; a, \bar{a})$ represents a desired stream function with a singularity at $\xi = a$.

Notation: If $\psi - Wc$, c an arbitrary constant, is regular at point a , the corresponding flow can be said to have a pseudo-vortex at infinity.

The functions A and B may be obtained in the following manner (see sec. 7 of reference 2):

Let a new variable be introduced:

$$\xi_1 = \xi - \xi_0 \quad (120)$$

Equation (86) then becomes

$$\frac{\partial^2 \psi}{\partial \xi_1 \partial \bar{\xi}_1} + F_m (\xi_1 + \xi_0, \bar{\xi}_1 + \bar{\xi}_0) = 0 \quad (121)$$

A fundamental solution of (121) with the affix at $\xi_1 = 0$ will be a fundamental solution of (86) with the affix at $\xi = \xi_0$.

Substituting $W = \frac{1}{2} a \log \xi_1 + \frac{1}{2} a \log \bar{\xi}_1 + B$, $a = a(\xi_1, \bar{\xi}_1)$, $B = B(\xi_1, \bar{\xi}_1)$ into (121) gives

$$\begin{aligned} \frac{1}{2} \left(a_{\xi_1} \bar{\xi}_1 + F_m a \right) \log \xi_1 + \frac{1}{2} \left(\frac{a_{\bar{\xi}_1}}{\bar{\xi}_1} + \frac{a_{\xi_1}}{\xi_1} \right) \\ + B_{\xi_1 \bar{\xi}_1} + F_m B = 0 \end{aligned} \quad (122)$$

Therefore a is a solution of (121) which has the property that $\left(a_{\bar{\xi}_1} / \bar{\xi}_1 + a_{\xi_1} / \xi_1 \right)$ is regular, at $\xi_1 = 0, \bar{\xi}_1 = 0$.

$$a = 1 - \int_0^{\xi_1} \int_0^{\bar{\xi}_1} F_m d\xi_2 d\bar{\xi}_2 + \int_0^{\xi_1} \int_0^{\bar{\xi}_1} F_m \left[\int_0^{\xi_2} \int_0^{\bar{\xi}_2} F_m d\xi_3 d\bar{\xi}_3 \right] d\xi_2 d\bar{\xi}_2 + \dots \quad (123)$$

is a desired solution of (121). And B is a solution of the equation

$$B_{\xi_1 \bar{\xi}_1} + F_m B + G = 0, \quad G = \left(a_{\bar{\xi}_1} / \xi_1 \right) + \left(a_{\xi_1} / \bar{\xi}_1 \right) \quad (124)$$

It follows that

$$B = \int_0^{\xi_1} \int_0^{\bar{\xi}_1} G d\xi_2 d\bar{\xi}_2 - \int_0^{\xi_1} \int_0^{\bar{\xi}_1} F_m \left[\int_0^{\xi_2} \int_0^{\bar{\xi}_2} G d\xi_3 d\bar{\xi}_3 \right] d\xi_2 d\bar{\xi}_2 + \dots \quad (125)$$

Remark 1: As indicated elsewhere, the theory of operators yields an alternative expression for a .

In references 2 and 12 a function ϵ was considered which is a solution of (75) and therefore which when substituted into (76) for E yields a solution of (121); ϵ has the following property:

$$\epsilon(\xi_1, \bar{\xi}_1, t) = 1 + \xi_1 \bar{\xi}_1 \epsilon_1(\xi_1, \bar{\xi}_1, t) \quad (126)$$

where ϵ_1 is again a regular function of $\xi_1, \bar{\xi}_1$. (See reference 12, formulas (1.12), (1.14), and (1.15).)

If the function E is denoted by $\epsilon(\xi_1, \bar{\xi}_1, t; \xi_0, \bar{\xi}_0)$ corresponding to equation (121), then

$$a(\xi_1, \bar{\xi}_1) = \int_{-1}^{+1} \epsilon dt / (1-t^2)^{1/2} = \pi/2 + \xi_1 \bar{\xi}_1 \int_{-1}^{+1} \epsilon_1 dt / (1-t^2)^{1/2}$$

yields a desired solution of (121). Thus

$$a(\xi, \bar{\xi}, \xi_0, \bar{\xi}_0) = \int_{-1}^{+1} \epsilon(\xi - \xi_0, \bar{\xi} - \bar{\xi}_0, t; \xi_0, \bar{\xi}_0) dt / (1-t^2)^{1/2} \quad (127)$$

The functions $Q^{(n)}(v/a_0)$ which correspond to ϵ are not real. This was the reason that in reference 3 and in the present paper a new solution E is introduced which yields real functions $Q^{(n)}(v/a_0)$.

In reference 12, sec. 1, the function ϵ is determined in a form of an infinite series.

It is observed that for various equations ϵ can be represented in a closed form.

The function W satisfies equation (86). Clearly

$$\frac{\partial W^*}{\partial \theta} = A_1 \frac{\theta - \theta_0}{[(\lambda - \lambda_0)^2 + (\theta - \theta_0)^2]^{1/2}} + A_2 \log [(\lambda - \lambda_0)^2 + (\theta - \theta_0)^2] + A_3 \quad (128)$$

where A_1, A_2 , and A_3 are entire functions, is also a solution of this equation

Notation: If $\psi = C_1 W^* - C_2 \frac{\partial W^*}{\partial \theta}$, C_1, C_2 arbitrary constants,

is regular at $\xi = a$, then it can be said that the corresponding flow has a combined pseudo-vortex and pseudo-doublet at infinity.

By refining this procedure (namely, considering functions $A_4(\xi - \xi_0)^{-1} + \bar{A}_4(\bar{\xi} - \bar{\xi}_0)^{-1} + A_5 + \bar{A}_5$ etc.)

other univalent solutions of (86), may be found which have singularities at point $\xi = a$.

Remark 2: In the case where the density pressure relation is of the form $p = A + \frac{\sigma}{\rho}$ the functions W^* and $\frac{\partial W^*}{\partial \theta}$ are

$$\frac{1}{2} \log [(\lambda - \lambda_0)^2 + (\theta - \beta)^2] \text{ and } \frac{\theta - \beta}{(\lambda - \lambda_0)^2 + (\theta - \beta)^2} \quad (129)$$

respectively, where $\lambda = \frac{1}{2} \log \left\{ \frac{[1 + (\gamma/a_0)^2]^{1/2} - 1}{[1 + (\gamma/a_0)^2]^{1/2} + 1} \right\}$. It is

noted that in this particular case new singularities are obtained by differentiating W^* with respect to λ . For instance,

$$W^{(01)} = \frac{\partial W^*}{\partial \lambda} = \frac{\lambda - \lambda_0}{(\lambda - \lambda_0)^2 + (\theta - \beta)^2} \quad (130)$$

is a singularity which is infinite of the first order and is independent of $\frac{\partial W^*}{\partial \theta}$.

14. The Passage to the Physical Plane

In the following a procedure will be described for determining the flow in the physical plane corresponding to a stream function, $\psi(v, \theta)$, given in the hodograph plane.

Considering $\frac{\partial x}{\partial \psi}$, $\frac{\partial y}{\partial \psi}$, $\frac{\partial x}{\partial \phi}$, $\frac{\partial y}{\partial \phi}$ as unknown, it is found from

$$\left. \begin{aligned} \frac{\partial \psi}{\partial x} \frac{\partial x}{\partial \psi} + \frac{\partial \psi}{\partial y} \frac{\partial y}{\partial \psi} &= \frac{\partial \psi}{\partial \psi} = 1, & \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial \phi} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial \phi} &= 1 \\ \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial \psi} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial \psi} &= \frac{\partial \phi}{\partial \psi} = 0, & \frac{\partial \psi}{\partial x} \frac{\partial x}{\partial \phi} + \frac{\partial \psi}{\partial y} \frac{\partial y}{\partial \phi} &= 0 \end{aligned} \right\} \quad (131)$$

that at every point at which the Jacobian

$$D = \frac{\partial(\phi, \psi)}{\partial(x, y)} = \frac{\partial \phi}{\partial x} \frac{\partial \psi}{\partial y} - \frac{\partial \phi}{\partial y} \frac{\partial \psi}{\partial x} = \frac{(v_1^2 + v_2^2) \rho}{\rho_0} = \frac{v^2}{\rho_0} \quad (132)$$

(see (24)) does not vanish and is finite, the relations

$$\left. \begin{aligned} D \frac{\partial x}{\partial \phi} &= \frac{\partial \psi}{\partial y}, & D \frac{\partial y}{\partial \phi} &= -\frac{\partial \psi}{\partial x} \\ D \frac{\partial x}{\partial \psi} &= -\frac{\partial \phi}{\partial y}, & D \frac{\partial y}{\partial \psi} &= \frac{\partial \phi}{\partial x} \end{aligned} \right\} \quad (133)$$

hold. Using (24) yields from (133) and (132)

$$\left. \begin{aligned} dx &= \frac{\partial x}{\partial \phi} d\phi + \frac{\partial x}{\partial \psi} d\psi = \frac{\cos \theta}{v} d\phi - \frac{\rho_0 \sin \theta}{\rho v} d\psi \\ dy &= \frac{\partial y}{\partial \phi} d\phi + \frac{\partial y}{\partial \psi} d\psi = \frac{\sin \theta}{v} d\phi + \frac{\rho_0 \cos \theta}{\rho v} d\psi \end{aligned} \right\} \quad (134)$$

Since by (30)

$$d\phi = \phi_v dv + \phi_\theta d\theta = -\frac{\rho_0(1-M^2)}{\rho v} \psi_\theta dv + \frac{v}{\rho} \rho_0 \psi_\theta d\theta \quad (135)$$

there is obtained

$$\begin{aligned} dx &= \frac{\rho_0}{\rho} \left\{ \left[-\frac{(1-M^2)\cos\theta}{v^2} \psi_\theta - \frac{\sin\theta}{v} \psi_v \right] dv + \left[\cos\theta \psi_v - \frac{\sin\theta}{v} \psi_\theta \right] d\theta \right\} \\ dy &= \frac{\rho_0}{\rho} \left\{ \left[-\frac{(1-M^2)\sin\theta}{v^2} \psi_\theta + \frac{\cos\theta}{v} \psi_v \right] dv + \left[\sin\theta \psi_v + \frac{\cos\theta}{v} \psi_\theta \right] d\theta \right\} \end{aligned} \quad (136)$$

or¹

$$\begin{aligned} dz &= dx + i dy = \frac{\rho_0}{\rho} e^{i\theta} \left\{ \left[-\frac{1-M^2}{v^2} \psi_\theta + i \frac{\psi_v}{v} \right] dv + \left[\psi_v + i \frac{\psi_\theta}{v} \right] d\theta \right\} \\ &= i \frac{\rho_0}{\rho} \frac{e^{i\theta}}{v} \left(\psi_\theta (\log v) + i \psi_v \right) (d(\log v) - i d\theta) + \frac{\rho_0}{\rho} \frac{e^{i\theta} M^2}{v^2} \psi_\theta dv \end{aligned} \quad (137)$$

Along a streamline, $\psi = \text{constant}$; that is, $\psi_v dv + \psi_\theta d\theta = 0$. Substituting $d\theta = -\psi_v dv / \psi_\theta$ into (137) gives for the streamlines in the physical plane the parametric representation:

$$\begin{aligned} x &= - \int \frac{\rho_0 \cos\theta}{\rho v^2} \frac{[(1-M^2)\psi_\theta^2 + v^2 \psi_v^2]}{\psi_\theta} dv \\ &= - \int \frac{\rho_0 \cos\theta}{\rho v^2} \frac{[\psi_\theta^2 + v^2 \psi_v^2]}{\psi_\theta} dv + \int \frac{\rho_0 \cos\theta M^2 \psi_\theta}{\rho v^2} dv \\ y &= - \int \frac{\rho_0 \sin\theta}{\rho v^2} \frac{[(1-M^2)\psi_\theta^2 + v^2 \psi_v^2]}{\psi_\theta} dv \\ &= - \int \frac{\rho_0 \sin\theta}{\rho v^2} \frac{[\psi_\theta^2 + v^2 \psi_v^2]}{\psi_\theta} dv + \int \frac{\rho_0 \sin\theta M^2 \psi_\theta}{\rho v^2} dv \end{aligned} \quad (138)$$

where the integration is carried out along a streamline, $\psi(v, \theta) = \text{constant}$. The integrals (138) represent a generalization of formulas (13). Substituting for ψ the expressions (69), (85), gives a parametric representation for the streamlines in terms of an arbitrary analytic function of one variable.

Suppose that the stream function $\psi(v, \theta)$ of a compressible fluid flow is given, where $\psi(v, \theta)$ is defined

¹It is assumed here and in the following that ψ_θ and ψ_v are univalent functions in the domain considered.

in the simply connected domain H (with boundary curve h) in the hodograph plane, and has a singularity at the point a (the image of $z = \infty$). (See, for instance, fig. 2b.)

Just as in the case of an incompressible fluid the conditions must be determined in order that the obstacle in the physical plane (which is formed by the image of the boundary curve h of H) be a closed curve. Clearly, the necessary and sufficient condition in order that the image of H in the physical plane be single-valued is that

$$\int_l \frac{\rho_0 e^{i\theta}}{\rho} \left\{ \left[-\frac{1-M^2}{v^2} \psi_\theta + i \frac{\psi_v}{v} \right] dv + \left[\psi_v + \frac{i\psi_\theta}{v} \right] d\theta \right\} = 0 \quad (139)$$

where l is any simple closed curve lying entirely in $H + h$.

Since the integrand of (139) is a complete differential, the value of the integral does not change if l is continuously deformed without leaving $H + h$ and without passing through the singular point a .

Thus, in particular, if the boundary curve h is chosen for l it follows that $d\psi = \frac{\partial\psi}{\partial v} dv + \frac{\partial\psi}{\partial\theta} d\theta = 0$, that is, $d\theta = - \left(\frac{\psi_v}{\psi_\theta} \right) d\theta$ along h , since h is a streamline. If this expression is inserted into (139), the condition is obtained that the image of h in the physical plane be a closed curve in a form analogous to (15) and (16).

On the other hand, the relation (139) can also be written in a different form which is often more suitable for applications.

In section 13 some standard types of single-valued singularities were introduced, that is, for every point $a = \alpha + i\beta$ functions were defined which are single-valued in the whole subsonic region and which satisfy equation (46) there, except at point a where they become infinite. Such functions are

$$W^{(0)}(v, \theta; \alpha, \beta) = W^*(\lambda, \theta; \lambda_0, \theta_0) \quad (140)$$

It is assumed here that the point $v = 0$ is not an interior point of domain H . If $v = 0$, $\theta = 0$ is an interior point it is well to proceed similarly but use as variables v_1 and v_2 instead of v and θ .

and

$$W^{(n)}(v, \theta; \alpha, \beta) = \frac{\partial^n W^{(0)}(\lambda, \theta; \lambda_0, \theta_0)}{\partial \theta^n} \quad (141)$$

where W^* is given by (119), $\beta = \theta_0$ and

$$\lambda_0 = \frac{1}{2} \log \left[\frac{1 - (1 - M_0^2)^{1/2}}{1 + (1 - M_0^2)^{1/2}} \left(\frac{1 + h(1 - M_0^2)^{1/2}}{1 - h(1 - M_0^2)^{1/2}} \right)^{1/h} \right] \quad (142)$$

$$M_0 = \frac{\alpha}{\left[a_0^2 - \frac{1}{2}(k-1)\alpha^2 \right]^{1/2}}$$

Notation: With every singularity $W^{(n)}$, $n = 0, 1, 2, \dots$ may be associated a (complex) number $X_n + iY_n$ which will be denoted by $R(W^{(n)})$.

$$R(W^{(n)}) = X_n + iY_n \quad (143)$$

$$= \int_c \frac{\rho_0}{\rho} e^{i\theta} \left\{ \left[-\frac{1 - M^2}{v^2} W_v^{(n)} + i \frac{W_v^{(n)}}{v} \right] dv + \left[W_v^{(n)} - i \frac{W_\theta^{(n)}}{v} \right] d\theta \right\}$$

where c is an arbitrary simple closed curve around a , the sense of integration being such that a always lies to the left.¹

Let it be assumed now that the stream function ψ can be represented in the neighborhood of a (the image of $z = \infty$) in the form

$$\psi = \sum_{n=1}^k A_n W^{(n)} + \Psi \quad (144)$$

where A_n are constants, and Ψ is a function which is regular at point a . Since curve c may be chosen for l , and since the integral (137) in which $W^{(n)}$ is replaced by

¹Since the integrand of (143) is a complete differential, the value of the integral is independent of the choice of the path of c integration.

Ψ vanishes, it is concluded that an alternate form of the condition that the image of h in the physical plane, for a flow the stream function of which is given by (144), be a closed curve is that

$$\sum_{n=1}^k A_n R(w^{(n)}) = 0 \quad (145)$$

It will be of interest to discuss in more detail the evaluation of the quantities $R(w^{(n)})$ in the special case, when $k = -1$.

As was indicated in section 13 in this case

$$w^{(0)} = \frac{1}{2} \log [(\lambda - \lambda_0)^2 + (\theta - \beta)^2], \quad w^{(1)} = \frac{\theta - \beta}{(\lambda - \lambda_0)^2 + (\theta - \beta)^2} \quad (146)$$

where $\alpha + i\beta = a$.

$$\lambda = \frac{1}{2} \log \left\{ \frac{[1 + (v/a_0)^2]^{1/2} - 1}{[1 + (v/a_0)^2]^{1/2} + 1} \right\}, \quad \lambda_0 = \frac{1}{2} \log \left\{ \frac{[1 + (\alpha/a_0)^2]^{1/2} - 1}{[1 + (\alpha/a_0)^2]^{1/2} + 1} \right\}$$

In addition to these singularities are obtained (in this particular case) by differentiating with respect to λ , the singularities

$$w_{(01)} = \frac{\partial w^{(0)}}{\partial \lambda} = \frac{\lambda - \lambda_0}{(\lambda - \lambda_0)^2 + (\theta - \beta)^2}, \text{ etc.} \quad (147)$$

Substituting

$$w_v^{(0)} = \frac{\lambda - \lambda_0}{[(\lambda - \lambda_0)^2 + (\theta - \beta)^2]^{1/2} [1 + (v/a_0)^2]^{1/2}}$$

$$w_\theta^{(0)} = \frac{\theta - \beta}{[(\lambda - \lambda_0)^2 + (\theta - \beta)^2]^{1/2}}$$

or

$$W_v^{(1)} = \frac{2(\lambda - \lambda_0)(\theta - \beta)}{[(\lambda - \lambda_0)^2 + (\theta - \beta)^2]^2 v [1 + (v/a_0)^2]^{1/2}}$$

$$W_\theta^{(1)} = \frac{(\lambda - \lambda_0)^2 - (\theta - \beta)^2}{[(\lambda - \lambda_0)^2 + (\theta - \beta)^2]^2} \quad (148)$$

or

$$W_v^{(01)} = \frac{-(\lambda - \lambda_0)^2 + (\theta - \beta)^2}{[(\lambda - \lambda_0)^2 + (\theta - \beta)^2]^2 v [1 + (v/a_0)^2]^{1/2}}$$

$$W_\theta^{(01)} = \frac{-2(\lambda - \lambda_0)(\theta - \beta)}{[(\lambda - \lambda_0)^2 + (\theta - \beta)^2]^2}$$

into (143) gives the corresponding values $R(W^{(k)}) = X_k + iY_k$, $k = 0, 1$, and $R(W^{(01)}) = X_{01} + iY_{01}$, respectively.

For instance,

$$X_0 = \int_c \frac{1}{[(\lambda - \lambda_0)^2 + (\theta - \beta)^2]} \left\{ \left[\frac{-\cos \theta (\theta - \beta)}{v^2 (1 + (v/a_0)^2)^{1/2}} - \frac{\sin \theta (\lambda - \lambda_0)}{v^2} \right] dv \right.$$

$$\left. + \left[\frac{\cos \theta (\lambda - \lambda_0)}{v} + \frac{(1 + (v/a_0)^2)^{1/2} (\theta - \beta) \sin \theta}{v} \right] d\theta \right\}$$

$$Y_0 = \int_c \frac{1}{[(\lambda - \lambda_0)^2 + (\theta - \beta)^2]} \left\{ \left[\frac{-\sin \theta (\theta - \beta)}{v^2 (1 + (v/a_0)^2)^{1/2}} + \frac{\cos \theta (\lambda - \lambda_0)}{v^2} \right] dv \right.$$

$$\left. + \left[\frac{\sin \theta (\lambda - \lambda_0)}{v} - \frac{(1 + (v/a_0)^2)^{1/2} (\theta - \beta) \cos \theta}{v} \right] d\theta \right\}$$

15. Appendix

I. Verification that the expressions (136) are complete differentials.—To show that dx as given by¹ (136a) is a

¹By (136a) will be denoted in this section the first expression of (136), and by (136b) the second expression.

complete differential, it is necessary to prove that the coefficients a and b

$$\left. \begin{aligned} a &= \rho_o \left[-\frac{\cos \theta (1 - M^2)}{\rho v^2} \psi_\theta - \frac{\sin \theta}{v} \psi_v \right] \\ b &= \rho_e \left[\frac{\cos \theta}{\rho} \psi_v - \frac{\sin \theta}{\rho v} \psi_\theta \right] \end{aligned} \right\} \quad (149)$$

of dv and $d\theta$ of the right-hand side of (136a) satisfy the relation

$$\frac{\partial a}{\partial \theta} = \frac{\partial b}{\partial v} \quad (150)$$

Recalling (30) yields

$$\frac{\partial \phi}{\partial \theta} = \frac{\rho_o v}{\rho} \frac{\partial \psi}{\partial v}, \quad \frac{\partial \phi}{\partial v} = -\frac{\rho_o (1 - M^2)}{\rho v} \frac{\partial \psi}{\partial \theta} \quad (151)$$

so that

$$\left. \begin{aligned} a &= \rho_o \left(\frac{\cos \theta}{v} \phi_v - \frac{\sin \theta}{\rho v} \psi_v \right) \\ b &= \rho_o \left(\frac{\cos \theta}{v} \phi_\theta - \frac{\sin \theta}{\rho v} \psi_\theta \right) \end{aligned} \right\} \quad (152)$$

Therefore

$$\frac{\partial a}{\partial \theta} = \rho_o \left(-\frac{\sin \theta}{v} \phi_v + \frac{\cos \theta}{v} \phi_{v\theta} - \frac{\cos \theta}{\rho v} \psi_v - \frac{\sin \theta}{\rho v} \psi_{v\theta} \right) \quad (153)$$

$$\begin{aligned} \frac{\partial b}{\partial v} &= \rho_o \left(-\frac{\cos \theta}{v^2} \phi_\theta + \frac{\cos \theta}{v} \phi_{\theta v} + \frac{\sin \theta}{\rho^2} \frac{\rho_v}{v} \psi_\theta \right. \\ &\quad \left. + \frac{\sin \theta}{\rho v^2} \psi_\theta - \frac{\sin \theta}{\rho v} \psi_{v\theta} \right) \quad (154) \end{aligned}$$

Thus it is necessary to prove that

$$\begin{aligned} \left(-\frac{\sin \theta}{v} \phi_v - \frac{\cos \theta}{\rho v} \psi_v \right) &= -\frac{\cos \theta}{v^2} \phi_\theta + \frac{\sin \theta}{\rho^2} \frac{\rho_v}{v} \psi_\theta \\ &\quad + \frac{\sin \theta}{\rho v^2} \psi_\theta \quad (155) \end{aligned}$$

If use is made of (30) again, it is necessary only to show that

$$\frac{1 - M^2}{\rho v^2} = \frac{\rho_v}{\rho^2 v} + \frac{1}{\rho v^2} \quad (156)$$

$$M^2 = -v \frac{\partial}{\partial v} (\log \rho) \quad (157)$$

But from (25) it is known that

$$\rho = \rho_0 \left[1 - \frac{k-1}{2a_0^2} v^2 \right]^{\frac{1}{k-1}} \quad (158)$$

so that

$$v \frac{\partial}{\partial v} (\log \rho) = - \frac{v^2}{a_0^2 - \frac{1}{2}(k-1)v^2}$$

If this is compared with (31) it is seen that this is exactly equal to $-M^2$ and it therefore has been verified that the right-hand side of (136a) is a complete differential. In a similar fashion it might be shown that the right-hand side of (136b) is also a complete differential. Hence, since this is the case, it follows that the integrals (138) are independent of the path of integration.

II. A proof of an auxiliary lemma.— In the following it will be proved that $F(2\lambda)$ (see sec. 9) can be approxi-

mated by polynomials $F_m(2\lambda) = \sum_{s=0}^m \alpha_s^{(m)} e^{2s\lambda}$, $\alpha_0^{(m)} = 0$

in every interval $(-\infty, \lambda_0)$, $\lambda_0 < 0$, and indicated how to determine the $F_m(2\lambda)$.

If 2λ is replaced by $\log X$, the $F[2\lambda(X)]$ is a continuous function of X in the interval $(0, X_0)$, $X_0 < 1$, and by classical results, it is obvious that it can be approximated by a polynomial in X . It will be seen that it will not suffice merely to approximate F , but in addition to this it will be possible to require that any given number of the derivatives of F be approximated in the interval $(-\infty, \lambda_0)$ by the corresponding derivatives of F_m .

It is, however, of interest to give a more explicit form of the approximating polynomials. This will make it possible to determine the corrections of $Q(n)$ obtained in section 12 which have to be made in order to obtain functions $Q(n)$ corresponding to a given F_m .

If M increases steadily from 0 to 1, λ increases from $-\infty$ to 0. Since the relation $M \rightarrow \lambda$ is a one-to-one correspondence, there corresponds to every λ_0 , $\lambda_0 < 0$, an $M_0 = M(\lambda_0) < 1$. Therefore, if $\lambda \in (-\infty, \lambda_0)$, then $M(\lambda) \in (0, M_0)$. For $M \leq M_0$, the function

$$F = \frac{(k+1) M^4 [-(3k-1) M^4 - 4(3-2k) M^2 + 16]}{64(1-M^2)^3}$$

may be approximated by a polynomial F_n of the $(2n+8)^{\text{th}}$ degree in M^2 ,

$$F_n(M) = \frac{(k+1)}{64} M^4 \left[-(3k-1) M^4 - 4(3-2k) M^2 + 16 \right] \left[\sum_{v=1}^n (v^{-3})(-1)^v M^{2v} \right] \quad (159)$$

Only a finite number of powers of M^2 appears in (149). It will now be shown that M^2 can be developed in the uniformly convergent series

$$M^2 = \sum_{k=1}^{\infty} \beta_k X^k, \quad X = 2 \left(\frac{(k+1)^{1/2} - (k-1)^{1/2}}{(k+1)^{1/2} + (k-1)^{1/2}} \right)^{\frac{k+1}{k-1}} e^{2\lambda}, \quad \lambda < 0 \quad (160)$$

Instead of considering M^2 , it is well to introduce

$$s = 1 - (1 - M^2)^{1/2} \quad (161)$$

Since $M^2 = 1 - (1 - s)^2$, it will suffice to determine the series for s .

From (48) follows

$$X = \frac{s}{2-s} \left\{ 2 \frac{(1+h^{-1}-s)(h^{-1}-1)^{1/2h}}{(h^{-1}-1+s)(1+h^{-1})} \right\}, \quad h = \left(\frac{k-1}{1+k} \right)^{1/2}, \quad k > 1 \quad (162)$$

For simplicity's sake it will be assumed in the following that

$$h \leq \frac{1}{2} \quad \text{that is, } k \leq 5/3 \quad (163)$$

Now consider the function $X = X(s)$ as a function of the complex variable s , and investigate its behavior in the domain $|s| < 1$.

Remark: $X(s) = \frac{s}{2-s} \left(\frac{1+h^{-1}-s}{h^{-1}-1+s} \right)^{\frac{1}{h}} + \frac{2H\pi i}{h}$ is a many-

valued function because any integer may be taken for H . Since, however, its branch points $s=1+h^{-1}$ and $s=h^{-1}-1$ are outside $|s| < 1$ it is necessary only to consider one of its branches. Therefore, $H=0$ is chosen, so that whenever $X(s)$ is mentioned this branch will be always understood.

In order to prove that the image of $|s| < 1$ is a schlicht domain in the X -plane it is noted at first that $X(s)$, s real, is a real function, and therefore the image will be a domain which is symmetric with respect to the real axis. The image of $s = +1$ will be the point 1, and the image of $s = -1$ will be a point of the negative real axis.

It will be shown that, if $\phi = \arg s$, varies from 0 to π , X increases steadily. Setting $s = e^{i\phi}$ yields

$$|X| = \frac{1}{(5-4 \cos \phi)^{1/2}} M^{1/2h}, \quad M = \frac{(1+h^{-1})^2 + 1 - 2(1+h^{-1}) \cos \phi}{(h^{-1}-1)^2 + 1 + 2(h^{-1}-1) \cos \phi} \quad (164)$$

and

$$\begin{aligned}
\frac{d|X|}{d\phi} &= -\frac{1}{2} \left(\frac{4 \sin \phi}{(5-4 \cos \phi)^{3/2}} \right) M^{1/2h} \\
&\quad + \left[\frac{1}{2h} \left(\frac{1}{(5-4 \cos \phi)^{1/2}} \right) M^{1/2h-1} \right] \\
&\quad \times \left[\frac{\left((1-h^{-1})^2 + 1 + 2(h^{-1}-1) \cos \phi \right) (1+h^{-1}) \sin \phi + 2(h^{-1}-1) 2 \sin \phi \left((1+h^{-1})^2 + 1 - 2(1+h^{-1}) \cos \phi \right)}{\left((1-h^{-1})^2 + 1 + 2(h^{-1}-1) \cos \phi \right)^2} \right] \\
&= \left(\frac{2 \sin \phi}{(5-4 \cos \phi)^{1/2}} \right) M^{1/2h-1} \frac{1}{(1-h^{-1})^2 + 1 + 2(h^{-1}-1) \cos \phi} P \quad (165) \\
P &= \left[- \left\{ \frac{(1+h^{-1})^2 + 1 - 2(1+h^{-1}) \cos \phi}{5 - 4 \cos \phi} \right\} \right. \\
&\quad \left. + \frac{h^{-4}}{(h^{-1}-1)^2 + 1 + 2(h^{-1}-1) \cos \phi} \right] \\
&= 4(h^{-2}-1) \left(\frac{(h^{-2}+1) - (h^{-2}+2) \cos \phi + \cos^2 \phi}{(5-4 \cos \phi) \left((h^{-1}-1)^2 + 1 + 2(h^{-1}-1) \cos \phi \right)} \right)
\end{aligned}$$

For all values $0 \leq \phi \leq \pi$ the expression (165) has the same sign as P . The denominator of P is always positive, and the numerator is positive for $-1 < \cos \phi < 1$. Clearly for all values of ϕ , $0 < \phi < \pi$, P , and therefore (165) is positive.

Thus the boundary curve of the image of $|s| < 1$ is a curve which does not intersect itself. By classical theorems of the theory of functions the domain bounded by this curve is schlicht. Clearly it includes in its interior the domain $|X| < 1$.

Since the image of $|s| < 1$ is schlicht and includes $|X| < 1$, the inverse function $s = s(X)$ is regular in $|X| < 1$ and by Cauchy's theorem can be expanded in $|X| < 1$

in the form of an infinite series, $s(X) = \sum_{v=1}^{\infty} a_v X^v$.

For every $X_0 < 1$ and every $\epsilon > 0$ there exists an N

such that $\left| s(X) - \sum_{v=1}^N \beta_v X^v \right| \leq \epsilon$ for $|X| < X_0$. Thus $\sum_{v=1}^N \beta_v X^v$

yields the required approximation.

Remark: Clearly N can be determined so large that any given number of derivatives of $\sum_{v=1}^N \beta_v X^v$ approximates the corresponding derivatives of $s(X)$.

It is noted further that a formal computation yields for the right-hand side of (152) for $|X| < 1$

$$\begin{aligned} X = s - \frac{1}{2}(2k+1)s^2 + \frac{1}{4}(4k^2+2k-1)s^3 \\ - \frac{1}{24}(24k^3+8k^2-14k-1)s^4 + \frac{1}{48}(48k^4+4k^3-44k^2+2k+5)s^5 \\ - \frac{1}{480}(480k^5-104k^4-572k^3+148k^2+126k-25)s^6 + \frac{1}{2880}(2880k^6 \\ -1584k^5-3944k^4+2212k^3+1140k^2-602k-5)s^7 + \dots \end{aligned} \quad (166)$$

The inverse function is

$$\begin{aligned} s = X + \frac{1}{2}(2k+1)X^2 + \frac{1}{4}(4k^2+6k+3)X^3 + \frac{1}{24}(24k^3 \\ +68k^2+76k+29)X^4 + \frac{1}{48}(48k^4+212k^3+392k^2+328k \\ +103)X^5 + \frac{1}{480}(480k^5+2976k^4+7968k^3 \\ +10788k^2+7266k+1935)X^6 + \frac{1}{2880}(2880k^6 \\ +23472k^5+84232k^4+162124k^3+173940k^2 \\ +98086k+22675)X^7 + \dots \end{aligned} \quad (167)$$

By the present result this series converges in $|X| < 1$, and therefore for $0 \leq X < 1$.

III. ADDITIONAL REMARKS

16. The Boundary Value Problem in the Physical Plane.

Mixed Flow

The theory developed in the second part of the paper leads to various methods for constructing flows around airfoils.

The primary problem to be faced in the theory of airfoils is to determine the flow with a certain velocity at infinity around an obstacle given in the physical plane. This leads to a very complicated nonlinear problem in the hodograph plane since the domain where the flow is defined is determined by the flow itself. However, this problem may be considerably simplified if it is agreed to obtain a flow around an obstacle which approximates the given obstacle.

The hodographs of flows of an incompressible fluid around profiles of certain types and for a number of angles of attack may be determined once and for all.

The present approach also makes it possible to construct functions satisfying (32) and having singularities of the kind required - that is, singularities of the flow of a compressible fluid which yield sources, vortices, and doublets.

A hodograph is chosen which in the case of an incompressible fluid leads to the desired profile.

Let $\Psi(v, \theta)$ be some solution of (32) which possesses the required singularity at point a (the image of $z = \infty$). A solution of $\Psi(v, \theta)$ of (32) is further determined, which is regular in the domain H and such that

$$\Psi(v, \theta) + \psi(v, \theta)$$

assumes a constant value on the boundary h of H .

The obtained function is a hodograph of a flow of a compressible fluid the image of which in the physical plane will in many instances not differ considerably from the given profile. This method of attack can be refined: By the foregoing procedure the initial profile is distorted in a certain way; if the given profile is distorted in opposite directions

and if the procedure described is repeated to the distorted profile, then in many instances a better approximation is obtained. This method may be repeated until the desired degree of accuracy is attained.¹

However, this procedure has the inconvenience that in order to determine ψ it is necessary (at each step) to solve a boundary value problem for the equation (32) which requires rather long computation. In another paper the author has developed in detail an alternative to this method, in which he avoids the necessity of solving boundary value problems.

In the present considerations attention was in the main directed toward the subsonic case. In addition to the method of attack, which is based on considerations of section 8 of the second part, there exists another possibility for handling the mixed problem - that is, to construct flows which are partially subsonic and partially supersonic.

17. The Representation of the Stream Function of a Subsonic Flow in the Region in Which the Velocity is Near the Velocity of Sound

Partially Supersonic Flow

In the region ($M \leq M_0 < 1$) where M_0 is near 1, the series (85) converges very slowly, and it is therefore necessary to employ a large number of terms in order to obtain a good approximation for ψ^* . If this be the case, it is then expedient to replace the expansion (85) by (103)².

This is, however, not the only way of overcoming this difficulty, and in the following, other means of so doing will be indicated; this alternate approach employs the method of "analytic continuation."

¹It may be observed that a similar procedure can be applied to prove that for every profile (satisfying certain conditions) there exists a flow of a compressible fluid.

²This method will be developed in more detail in a future report of the author.

Let $\psi(v, \theta)$ be determined in a domain, say H , and let $\{\psi_n\}$, $n = 1, 2, \dots$, be a "complete" system of particular solutions of (46), each ψ_n being determined in a domain G . Suppose that H and G actually do overlap and denote their common part by I . Further, let $\sum_{n=1}^{\infty} a_n \psi_n$ be the series expansion of ψ in I . Frequently $\sum_{n=1}^{\infty} a_n \psi_n$ will converge outside of I , say in the domain $H_2 - I$, where H_2 is G or some part of it. If, in addition, $\sum_{n=1}^{\infty} a_n \psi_n$ can be termwise differentiated twice in H_2 , it represents the analytic continuation of ψ in $H_2 - I$.

Remark: The requirement that $\sum_{n=1}^{\infty} a_n \psi_n$ coincide with ψ in a domain I , can be replaced by another requirement, which will be explained later.

Frequently, the domain H_2 in which the stream function can be represented in the form $\sum_{n=1}^{\infty} a_n \psi_n$ covers a supersonic region as well, and consequently this method will then yield the flow in this latter region. In this manner, a method (based on considerations other than those of sec. 8) for determining a mixed flow may be obtained.

Two alternate forms of this method will be discussed in the following.

First Method

In order to develop the first approach, an auxiliary lemma must first be proved.

Lemma: Let $p(v, \theta)$, $[v_0 \leq v \leq v_1, -L \leq \theta \leq L]$ be an analytic function of two real variables v, θ , and let

$$\sum_{v=0}^{\infty} a_v(v) \cos \frac{\pi v \theta}{L} + b_v(v) \sin \frac{\pi v \theta}{L}$$

$$a_0(v) = \frac{1}{2} L^{-1} \int_{-L}^L p(v, \theta) d\theta, \quad a_v(v) = \frac{1}{L} \int_{-L}^L p(v, \theta) \cos \frac{\pi v \theta}{L} d\theta,$$

$$b_v(v) = \frac{1}{L} \int_{-L}^L p(v, \theta) \sin \frac{\pi v \theta}{L} d\theta \quad (168)$$

be its Fourier development. The series (168) converges uniformly and can be differentiated termwise any finite number of times both with respect to v and with respect to θ .

Proof: Let

$$\frac{\partial^k p(v, \theta)}{\partial v^k} \sim \sum_v \left[\frac{d^k a_v(v)}{dv^k} \cos \frac{\pi v \theta}{L} + \frac{d^k b_v(v)}{dv^k} \sin \frac{\pi v \theta}{L} \right], \quad (k = 1, 2)$$

Now, since p is an analytic function of v and θ , $\frac{\partial^2 p}{\partial v \partial \theta}$ is also an analytic function, and therefore

$$\int_{-L}^L \left(\frac{\partial^2 p}{\partial v \partial \theta} \right)^2 d\theta \leq A \quad (169)$$

is bounded, uniformly, in v :

On the other hand,

$$\sum_{v=m}^M \left(\left| \frac{da_v(v)}{dv} \right| + \left| \frac{db_v(v)}{dv} \right| \right) \leq \left[\left(\sum_{v=m}^M \frac{2}{v^2} \right) \left(\sum_{v=m}^M v^2 \left(\frac{da_v}{dv} \right)^2 + v^2 \left(\frac{db_v}{dv} \right)^2 \right) \right]^{\frac{1}{2}} \leq \left[\sum_{v=m}^M \frac{2}{v^2} A \right]^{\frac{1}{2}} \quad (170)$$

from which the uniform convergence of the series

$$\sum_{v=0}^{\infty} \frac{da_v(v)}{dv} \cos \frac{\pi v \theta}{L} + \frac{db_v(v)}{dv} \sin \frac{\pi v \theta}{L} \quad (171)$$

follows. But (171) is the series which is obtained by differentiating (168) term by term. In a similar way the other cases may be handled.

Since every solution of an elliptic equation with analytic coefficients is an analytic function of two real variables, the result obtained can be applied to the case where $p(v, \theta)$ is the stream function $\psi(v, \theta)$ of a subsonic flow. Thus

$$\left. \begin{aligned} \sum_{v=0}^{\infty} \left[a_v(v) \cos \frac{\pi v \theta}{L} + b_v(v) \sin \frac{\pi v \theta}{L} \right] \\ a_0(v) = \frac{1}{2L} \int_{-L}^L \psi(v, \theta) d\theta, \\ a_v(v) = \frac{1}{L} \int_{-L}^L \psi(v, \theta) \cos \frac{\pi v \theta}{L} d\theta, \\ b_v(v) = \frac{1}{L} \int_{-L}^L \psi(v, \theta) \sin \frac{\pi v \theta}{L} d\theta, \quad (v = 1, 2, \dots) \end{aligned} \right\} \quad (172)$$

can be differentiated termwise. If now, following Chaplygin the author introduces instead of v the variable

$$\tau = \frac{v^2}{2a_0^2}$$

then the equation for ψ assumes the form

$$\frac{\partial}{\partial \tau} \left\{ 2\tau(1-\tau)^{-\beta} \frac{\partial \psi}{\partial \tau} \right\} + \frac{1 - (2\beta + 1)\tau}{2\tau(1-\tau)} (1-\tau)^{-\beta} \frac{\partial^2 \psi}{\partial \theta^2} = 0 \quad (173)$$

where $\beta = \frac{1}{(k-1)}$ (reference 1, p. 5, formula (12)).

Differentiating termwise gives

$$\sum_{v=0}^{\infty} \left[\left\{ \frac{d}{d\tau} \left(\tau(1-\tau)^{-\beta} \frac{da_v}{d\tau} \right) - \frac{1 - (2\beta + 1)\tau}{\tau(1-\tau)} (1-\tau)^{-\beta} \frac{v^2 \pi^2 a_v}{4L^2} \right\} \cos \frac{v\pi\theta}{L} + \left\{ \frac{d}{d\tau} \left(\tau(1-\tau)^{-\beta} \frac{db_v}{d\tau} \right) - \frac{1 - (2\beta + 1)\tau}{\tau(1-\tau)} (1-\tau)^{-\beta} \frac{\pi^2 v^2}{4L^2} b_v \right\} \sin \frac{v\pi\theta}{L} \right] = 0 \quad (174)$$

and, therefore, the a_v and b_v are each solutions of equation

$$\left. \begin{aligned} \frac{d}{d\tau} \left\{ \tau(1-\tau)^{-\beta} \frac{da_v}{d\tau} \right\} - \frac{1 - (2\beta + 1)\tau}{\tau(1-\tau)} (1-\tau)^{-\beta} \frac{v^2 \pi^2 a_v}{4L^2} &= 0 \\ \frac{d}{d\tau} \left(\tau(1-\tau)^{-\beta} \frac{db_v}{d\tau} \right) - \frac{1 - (2\beta + 1)\tau}{\tau(1-\tau)} (1-\tau)^{-\beta} \frac{\pi^2 v^2}{4L^2} b_v &= 0 \end{aligned} \right\} \quad (175)$$

(175) is a hypergeometric series and thus every solution of (175) may be written in the form

$$(A_v F_v + B_v F_v^*) \tau^{\frac{v}{2L}} \quad (176)$$

where A_v and B_v are constants and

$$\left. \begin{aligned} F_v &= F(\alpha_v, \beta_v; -\beta; 1 - \tau), \\ F_v^* &= (1 - \tau)^{-\beta+1} F(\gamma_v - \alpha_v, \gamma_v - \beta_v; 2 + \beta; 1 - \tau) \end{aligned} \right\} \quad (177)$$

$F(\alpha, \beta, \gamma, \tau)$ being the hypergeometric series. Here

$$\left. \begin{aligned} \gamma_v &= \left(\frac{v}{L} + 1 \right), \quad \beta = \frac{1}{k - 1} \\ \alpha_v &= \frac{1}{2} \left[\left(\frac{v}{L} - \beta \right) + \Delta_v \right], \quad \beta_v = \frac{1}{2} \left[\left(\frac{v}{L} - \beta \right) - \Delta_v \right] \\ \Delta_v &= \left[\left(\frac{v}{2L} \right)^2 (2\beta + 1) + \beta^2 \right]^{\frac{1}{2}} \end{aligned} \right\} \quad (178)$$

In order to determine the constants A_v, B_v , the following theorem is employed:

Let $\psi^{(1)}(v, \theta)$ and $\psi^{(2)}(v, \theta)$ [$v_0 \leq v \leq v_1, -L \leq \theta \leq L$] be solutions of an equation of elliptic type. If, along a line, say ($v = v_0$)

$$\psi^{(1)}(v_0, \theta) = \psi^{(2)}(v_0, \theta) \quad (179)$$

and

$$\left. \frac{\partial \psi^{(1)}(v, \theta)}{\partial v} \right|_{v=v_0} = \left. \frac{\partial \psi^{(2)}(v, \theta)}{\partial v} \right|_{v=v_0} \quad (180)$$

then in the whole domain [$v_0 \leq v \leq v_1, -L \leq \theta \leq L$]

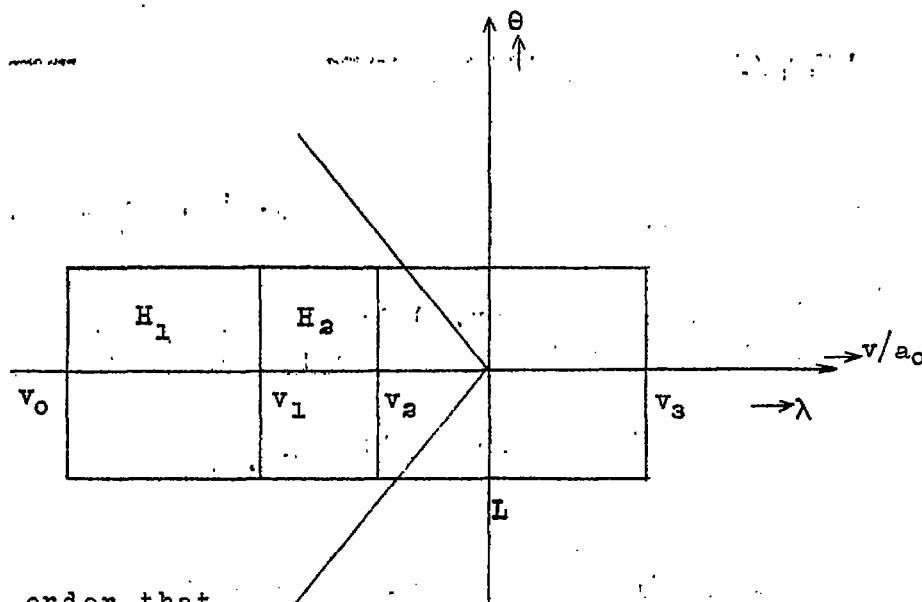
$$\psi^{(1)}(v, \theta) = \psi^{(2)}(v, \theta) \quad (181)$$

Remark: Suppose that the function $g(\xi)$, $\xi = \lambda - i\theta$, $\lambda = \lambda(v)$ (see (85)) is regular in some domain $H_1 + H_2$.

$$H_1 = [v_0 \leq v \leq v_1, -L \leq \theta \leq L]$$

$$H_2 = [v_1 \leq v \leq v_2, -L \leq \theta \leq L]$$

which domain lies in $[\theta^2 < 3\lambda^2, \lambda < 0]$. Then by the main theorem it follows that $\psi = \text{Im} P(g)$ is also regular in $H_1 + H_2$. Suppose that ψ has been evaluated in the domain H_1 , but it is desired to avoid the evaluation of ψ by means of (85) since this series converges very slowly in H_2 .



In order that

$$\sum_{v=1}^{\infty} s_v^{(1)}(\tau) \cos \frac{\pi v \theta}{L} + s_v^{(2)}(\tau) \sin \frac{\pi v \theta}{L} \quad (182)$$

$$s_v^{(k)}(\tau) = \left[A_v^{(k)} F_v + B_v^{(k)} F_v^* \right] \tau^{\frac{vL}{2}}, \quad (k = 1, 2) \quad (183)$$

be the solution ψ under consideration, the constants $A_v^{(k)}$, $B_v^{(k)}$ must be determined so that

$$\left. \begin{aligned} S_v^{(1)}(\tau) \Big|_{v=v_0} &= a_v(v_0) \\ S_v^{(2)}(\tau) \Big|_{v=v_0} &= b_v(v_0) \end{aligned} \right\} \quad (184)$$

$$\left. \begin{aligned} \frac{ds_v^{(1)}(\tau)}{dv} \Big|_{v=v_0} &= \frac{da_v(v)}{dv} \Big|_{v=v_0} \\ \frac{ds_v^{(2)}(\tau)}{dv} \Big|_{v=v_0} &= \frac{db_v(v)}{dv} \Big|_{v=v_0} \end{aligned} \right\} \quad (185)$$

It is noticed that

$$\begin{aligned} \frac{dS_v^{(k)}}{d\tau} &= \left[A_v^{(k)} \frac{dF_v}{d\tau} + B_v^{(k)} \frac{dF_v^*}{d\tau} \right] \tau^{\frac{v}{2L}} \\ &\quad + \frac{v}{2L} \tau^{\frac{v}{2L}-1} \left[A_v^{(k)} F_v + B_v^{(k)} F_v^* \right] \end{aligned} \quad (186)$$

$$\frac{dF_v}{d(1-\tau)} = \frac{\alpha_v \beta_v}{-\beta} F(\alpha_v + 1, \beta_v + 1; -\beta + 1; 1 - \tau)$$

$$\begin{aligned} \frac{dF_v^*}{d(1-\tau)} &= (1 + \beta)(1 - \tau)^{\beta} F(\gamma_v - \alpha_v, \gamma_v - \beta_v; 2 + \beta; 1 - \tau) \\ &\quad + (1 - \tau)^{\beta+1} \frac{(\gamma_v - \alpha_v)(\gamma_v - \beta_v)}{2 + \beta} F(\gamma_v - \alpha_v + 1, \gamma_v - \beta_v + 1; 3 + \beta; 1 - \tau) \end{aligned}$$

$$\gamma_v - \beta_v + 1; 3 + \beta; 1 - \tau)$$

Since in $\left[v_1 \leq v \leq v_2, -L \leq \theta \leq L \right]$, $\psi(v, \theta)$ is an analytic function, the series

$$\sum_{v=1}^{\infty} \left\{ \left[A_v^{(1)} F_v + B_v^{(1)} F_v^* \right] \frac{vL}{2} \cos \frac{\pi v \theta}{L} + \left[A_v^{(2)} F_v + B_v^{(2)} F_v^* \right] \frac{vL}{2} \sin \frac{\pi v \theta}{L} \right\} \quad (187)$$

and its derivatives converge uniformly and absolutely in this domain. Then (187) represents the solution ψ under consideration in the region H_2 . Moreover, this series (and its derivatives) may also converge outside of H_2 , say in $H_3 = [v_2 \leq v \leq v_3, -L \leq \theta \leq L]$. If H_3 partially lies outside of the domain $[\theta^2 < 3\lambda^2, \lambda < 0]$ (see sec. 11) then the obtained expression gives the analytic continuation of the solution outside of the domain of representation by the integral formulas (85). In particular, H_3 may include some region which lies in $M > 1$.

Very often it is known that the region, say L , where the velocity is supersonic is small. Now, instead of summing to infinity, take

$$\sum_{v=1}^N \left[S_v^{(1)}(\tau) \cos \frac{v\pi\theta}{L} + S_v^{(2)}(\tau) \sin \frac{v\pi\theta}{L} \right] \quad (188)$$

(see (183) and (177)) where N is sufficiently large; then (188) can be considered a sufficiently good approximation for analytic continuation of the stream function ψ under consideration. On the other hand, (188) represents ψ in the whole plane and therefore is particular in L .

In this way are obtained approximate flow patterns which are partially supersonic. In applying this method, it is necessary, however, to check whether the streamlines in L approach to smooth limit lines when m increases.

Second Method

In reference 2 the author has introduced different methods for computing sets of particular solutions¹, $\psi_n(v, \theta)$, of (32). (See p. 17 and p. 23 of reference 2.) The functions of each of these sets are defined for the subsonic and the supersonic range.

Let H be a domain in which it is desired to determine a hodograph with a supersonic velocity. Then H is divided into two overlapping parts H_1 and H_2 . In H_1 the velocity is throughout subsonic. The intersection of H_1 and H_2 is denoted by I . In figure 5, H_1 is that part of H for which $v \leq v_1$, and H_2 is that part for which $v \geq v_0$, $v_0 < v_1 < 1$. There is determined a function $\psi_0(\theta, v)$ which is defined in H_1 and has at point a , a prescribed singularity, and on the part of h which lies in $v < v_1$, h_1 , approximately constant values.

Now consider the functions

$$\psi^{(1)} = \psi_0(v, \theta) + \sum_{v=1}^n \alpha_v \psi_v(v, \theta)$$

$$\psi^{(2)} = \sum_{v=1}^m \beta_v \psi_v(v, \theta)$$

and determine the α_v and β_v in such a way that

$$\int_{h_1} |\psi_0 + \sum_{v=1}^n \alpha_v \psi_v|^2 ds + \int_{h_2} |\sum_{v=1}^n \beta_v \psi_v|^2 ds$$

$$+ \int_I |\psi_0 + \sum_{v=1}^n \alpha_v \psi_v - \sum_{v=1}^n \beta_v \psi_v|^2 dv d\theta$$

will be a minimum.

¹It is observed that it is possible also to use the Chaplygin solutions. See, for instance, reference 2, pp. 18-22.

If the boundary value problem has a solution (possessing certain properties on the boundary h) and if the system $\{\psi_v\}$ is complete, then it is possible to show (under certain additional conditions) that the limit function obtained by this process will yield the solution.

18. A Remark Concerning the Application of the Hodograph Method in the Three-Dimensional Case

The method developed in this paper yields a general formula for the stream functions of possible compressible fluid flow patterns.

As indicated in reference 2 (secs. 6 to 8) there exist other methods of obtaining particular solutions of equation (32); and for deriving from them solutions of (117). They often are not very convenient for practical purposes, and in many instances represent a flow only in a part of its domain of definition.

In the following will be indicated a method of obtaining particular solutions of equation (32) which has the disadvantages indicated but which can also be applied in the three-dimensional case.

As is well known, the velocity $\vec{q} = (u, v)$ of an irrotational fluid flow satisfies the equations

$$\nabla(\rho \vec{q}) = 0, \quad \nabla \times \vec{q} = 0, \quad (\text{Cauchy-Riemann equation}) \quad (189)$$

This suggests considering three-dimensional flows where the velocity $\vec{q} = (-u, -v, -w)$ satisfies the equation

$$\nabla(\rho \vec{q}) = 0, \quad \nabla \times \vec{q} = 0 \quad (190)$$

$\rho = \rho(V)$ being function of $V = (u^2 + v^2 + w^2)^{\frac{1}{2}}$ alone.

It follows from the second equation of (190) that there exists a potential ϕ , such that

$$\vec{q} = -\nabla\phi \quad (191)$$

Inserting this value in the first equation of (190) yields

$$\frac{\partial(\rho \partial\phi/\partial x)}{\partial x} + \frac{\partial(\rho \partial\phi/\partial y)}{\partial y} + \frac{\partial(\rho \partial\phi/\partial z)}{\partial z} = 0 \quad (192)$$

(192) is a very complicated nonlinear partial differential equation.

The introduction of u, v, w as new variables leads to a much simpler nonlinear differential equation.

Introduce as new variables

$$u = \frac{\partial\phi}{\partial x}, \quad v = \frac{\partial\phi}{\partial y}, \quad w = \frac{\partial\phi}{\partial z} \quad (193)$$

and as the new unknown function

$$\lambda = xu + yv + zw - \phi \quad (194)$$

Use (193) to obtain from (192)

$$\begin{aligned} \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} &= \left[\rho_u u + \rho \right] \frac{\partial u}{\partial x} + \rho_v u \frac{\partial v}{\partial x} + \rho_w u \frac{\partial w}{\partial x} \\ &+ \rho_u v \frac{\partial u}{\partial y} + \left[\rho_v v + \rho \right] \frac{\partial v}{\partial y} + \rho_w v \frac{\partial w}{\partial y} + \rho_u w \frac{\partial u}{\partial z} + \rho_v w \frac{\partial v}{\partial z} \\ &+ \left[\rho_w w + \rho \right] \frac{\partial w}{\partial z} = 0 \end{aligned} \quad (195)$$

It follows from (194) that

$$\left. \begin{aligned} \lambda_u &= x + u \frac{\partial x}{\partial u} + v \frac{\partial y}{\partial u} + w \frac{\partial z}{\partial u} - \frac{\partial\phi}{\partial x} \frac{\partial x}{\partial u} - \frac{\partial\phi}{\partial y} \frac{\partial y}{\partial u} - \frac{\partial\phi}{\partial z} \frac{\partial z}{\partial u} = x \\ \lambda_v &= y, \quad \lambda_w = z \end{aligned} \right\} \quad (196)$$

and

$$\left. \begin{aligned} \frac{\partial x}{\partial u} &= \lambda_{uu}, \quad \frac{\partial x}{\partial v} = \lambda_{uv}, \quad \frac{\partial x}{\partial w} = \lambda_{uw}, \quad \frac{\partial y}{\partial u} = \lambda_{vu}, \quad \frac{\partial y}{\partial v} = \lambda_{vv}, \\ \frac{\partial y}{\partial w} &= \lambda_{vw}, \quad \frac{\partial z}{\partial u} = \lambda_{uw}, \quad \frac{\partial z}{\partial v} = \lambda_{vw}, \quad \frac{\partial z}{\partial w} = \lambda_{ww}, \end{aligned} \right\} \quad (197)$$

From

$$\left. \begin{aligned} \frac{\partial u}{\partial x} &\equiv \frac{\partial u}{\partial x} \lambda_{uu} + \frac{\partial u}{\partial y} \lambda_{uv} + \frac{\partial u}{\partial z} \lambda_{uw} = 1 \\ \frac{\partial u}{\partial v} &\equiv \frac{\partial u}{\partial x} \lambda_{uv} + \frac{\partial u}{\partial y} \lambda_{vv} + \frac{\partial u}{\partial z} \lambda_{vw} = 0 \\ \frac{\partial u}{\partial w} &\equiv \frac{\partial u}{\partial x} \lambda_{uw} + \frac{\partial u}{\partial y} \lambda_{vw} + \frac{\partial u}{\partial z} \lambda_{ww} = 0 \end{aligned} \right\} \quad (198)$$

there is obtained

$$\frac{\partial u}{\partial x} = \frac{\begin{vmatrix} \lambda_{vv} & \lambda_{vw} \\ \lambda_{vw} & \lambda_{ww} \end{vmatrix}}{D}, \text{ and so forth} \quad (199)$$

where D denotes the determinant

$$D = \begin{vmatrix} \lambda_{uu} & \lambda_{uv} & \lambda_{uw} \\ \lambda_{uv} & \lambda_{vv} & \lambda_{vw} \\ \lambda_{uw} & \lambda_{vw} & \lambda_{ww} \end{vmatrix}$$

Substituting the values obtained in (199) into (195) yields the following equation for λ :

$$\left. \begin{aligned}
 & \left[\rho_u u + \rho \right] \begin{vmatrix} \lambda_{vv} & \lambda_{vw} \\ \lambda_{vw} & \lambda_{ww} \end{vmatrix} - \rho_v u \begin{vmatrix} \lambda_{uv} & \lambda_{vw} \\ \lambda_{uw} & \lambda_{ww} \end{vmatrix} + \rho_w u \begin{vmatrix} \lambda_{uv} & \lambda_{vv} \\ \lambda_{uw} & \lambda_{vw} \end{vmatrix} \\
 & - \rho_u v \begin{vmatrix} \lambda_{uv} & \lambda_{uw} \\ \lambda_{vw} & \lambda_{ww} \end{vmatrix} + \left[\rho_v v + \rho \right] \begin{vmatrix} \lambda_{uu} & \lambda_{uw} \\ \lambda_{wu} & \lambda_{ww} \end{vmatrix} - \rho_w v \begin{vmatrix} \lambda_{uu} & \lambda_{uv} \\ \lambda_{uv} & \lambda_{vv} \end{vmatrix} \\
 & + \rho_u w \begin{vmatrix} \lambda_{uv} & \lambda_{uw} \\ \lambda_{vv} & \lambda_{vw} \end{vmatrix} - \rho_v w \begin{vmatrix} \lambda_{uu} & \lambda_{uw} \\ \lambda_{uv} & \lambda_{ww} \end{vmatrix} + (\rho_w w + \rho) \begin{vmatrix} \lambda_{uu} & \lambda_{uv} \\ \lambda_{uv} & \lambda_{vv} \end{vmatrix} = 0
 \end{aligned} \right\} (200)$$

Here $\rho = \rho(u^2 + v^2 + w^2)^{\frac{1}{2}}$ is a known function.

There now arises the problem of determining particular solutions of (200). Clearly, this can be done by using the series developments

$$\sum_{m,n,p} A_{mnp} u^m v^n w^p \quad (201)$$

which satisfy equation (200).

Such a series development which represents (in the hodograph space) the potential function ϕ of a possible flow pattern of a compressible fluid converges only in the neighborhood of the origin.

However, there exist methods of determining ϕ , in the whole region of the real (u,v,w) space where ϕ is regular. Such a representation, for instance, is given in many cases by

$$\phi(x,y,z) = \lim_{k \rightarrow 0} \sum_{m,n,p} \frac{A_{mnp} u^m v^n w^p}{\Gamma[1 + k(m + n + p)]}$$

CONCLUDING REMARKS

The main result of the present report consists in deriving a formula which transforms an arbitrary analytic function of a complex variable into a stream function of a compressible subsonic flow.

This formula yields compressible flows around symmetric (and certain nonsymmetric) obstacles.

The main difficulty arises in adapting the formula to a given shape of the obstacle. Approximate methods for solving this problem are indicated in section 16.

Since all expressions appearing in the theory of a compressible fluid flow are much more complicated than those occurring in the study of incompressible flows, a careful investigation of the numerical methods to be applied is necessary.

A considerable part of the numerical work consists in preparing tables of auxiliary functions such as $Q^{(n)}$, which have to be used in all particular cases. In this paper the functions $Q^{(n)}$ are computed up to $n = 4$, for $k = 1.4$. Tables for the $Q^{(n)}$'s for higher values of the superscript n will be necessary if flows with maximum Mach number approaching 1 are to be considered.

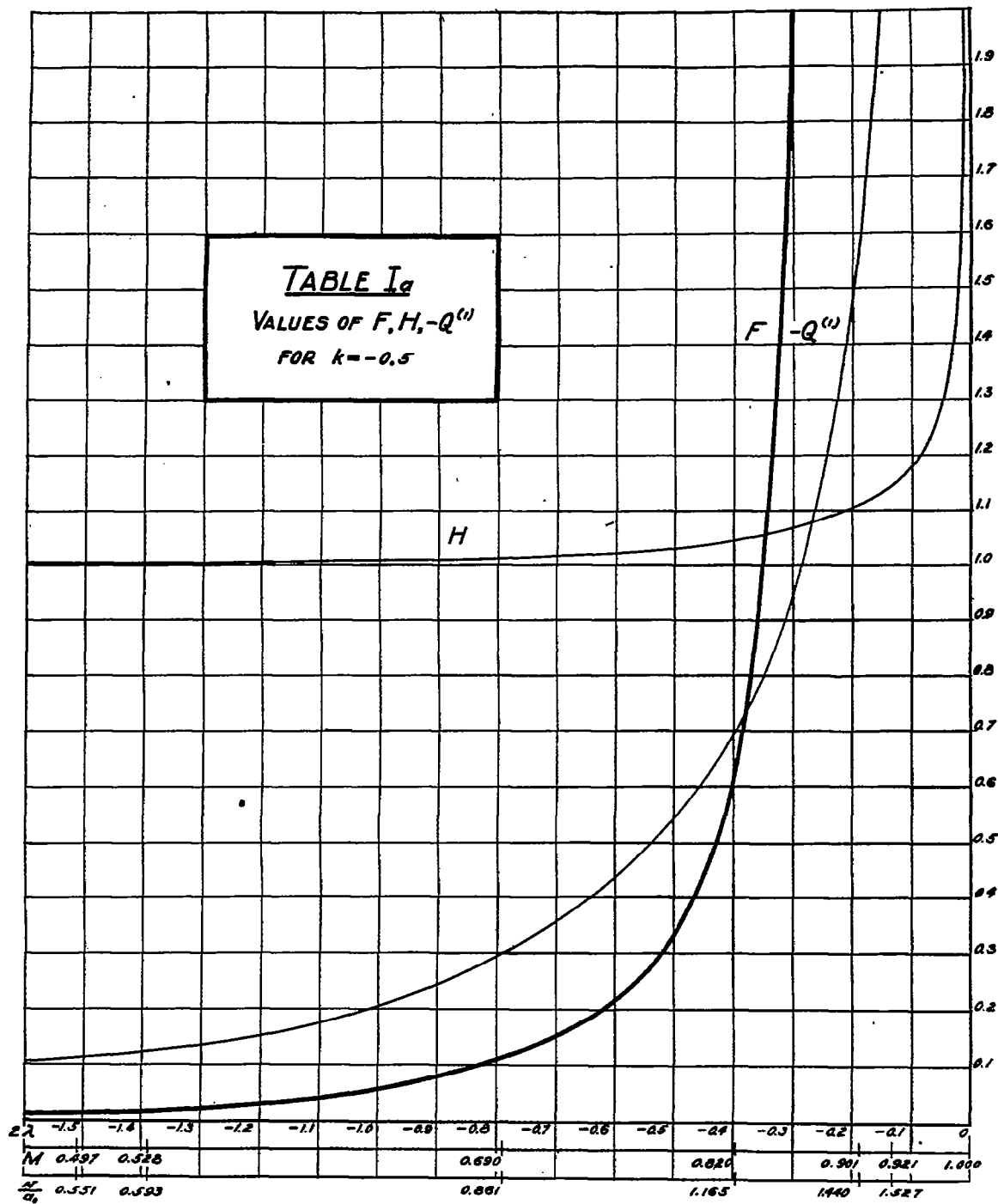
Each particular problem also involves the performance of certain integration processes. In order to advance the application of this theory it would be necessary to use efficient modern computing devices.

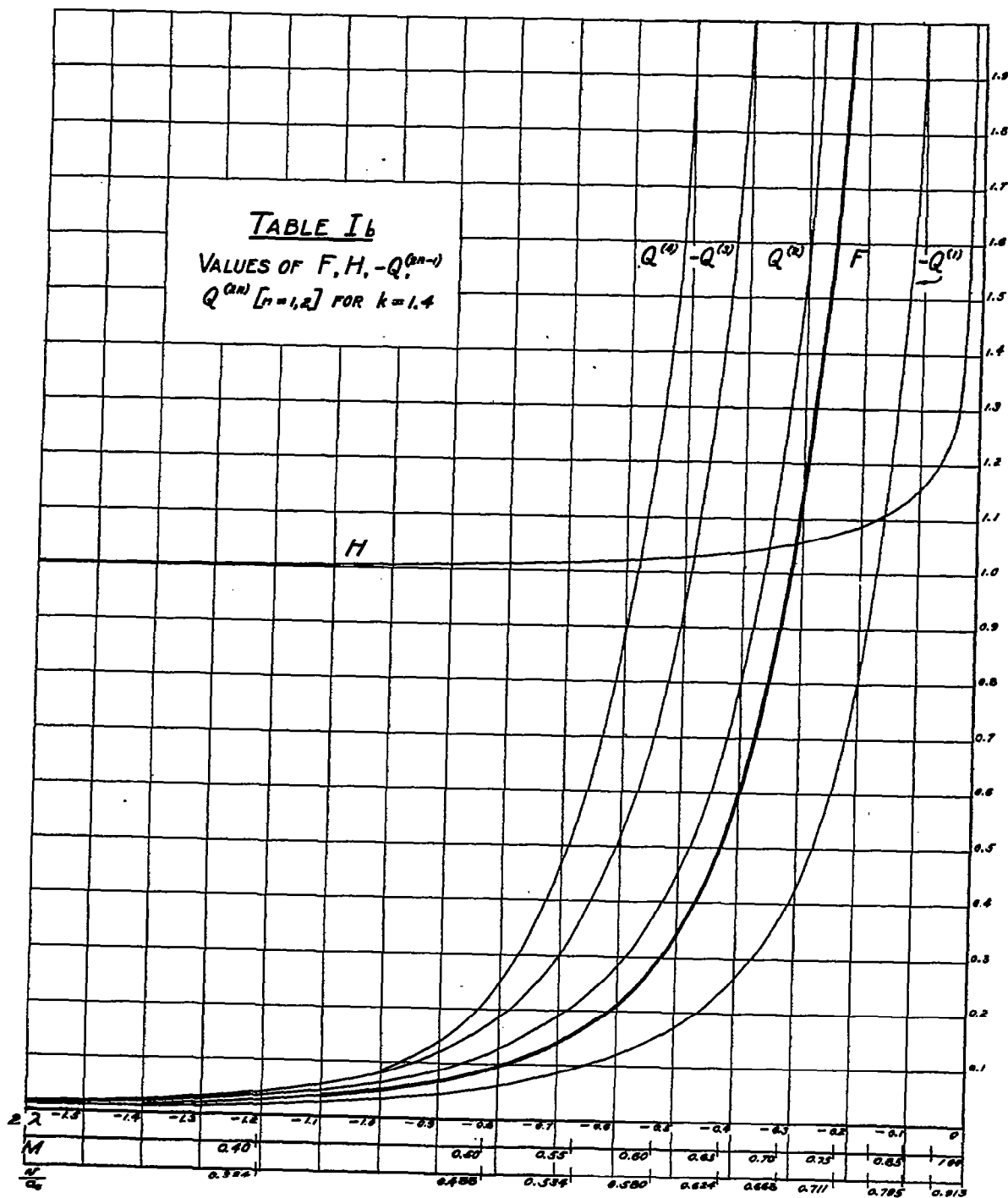
The present paper deals only with subsonic flows. It should be emphasized that the development of the theory will permit consideration of flows for which the maximal velocity exceeds that of sound. (See sec. 17.)

Brown University,
Providence, R. I., May 15, 1944.

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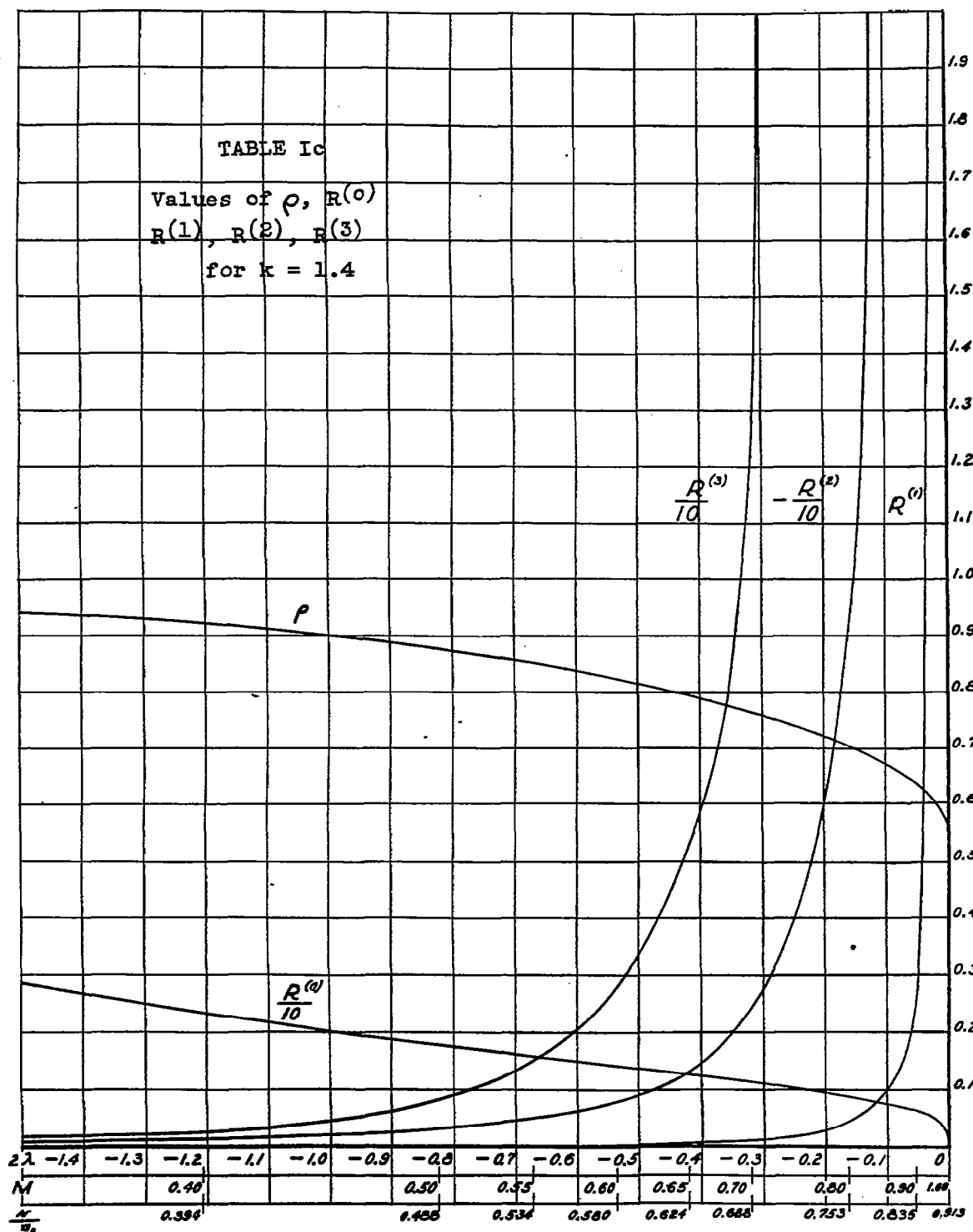
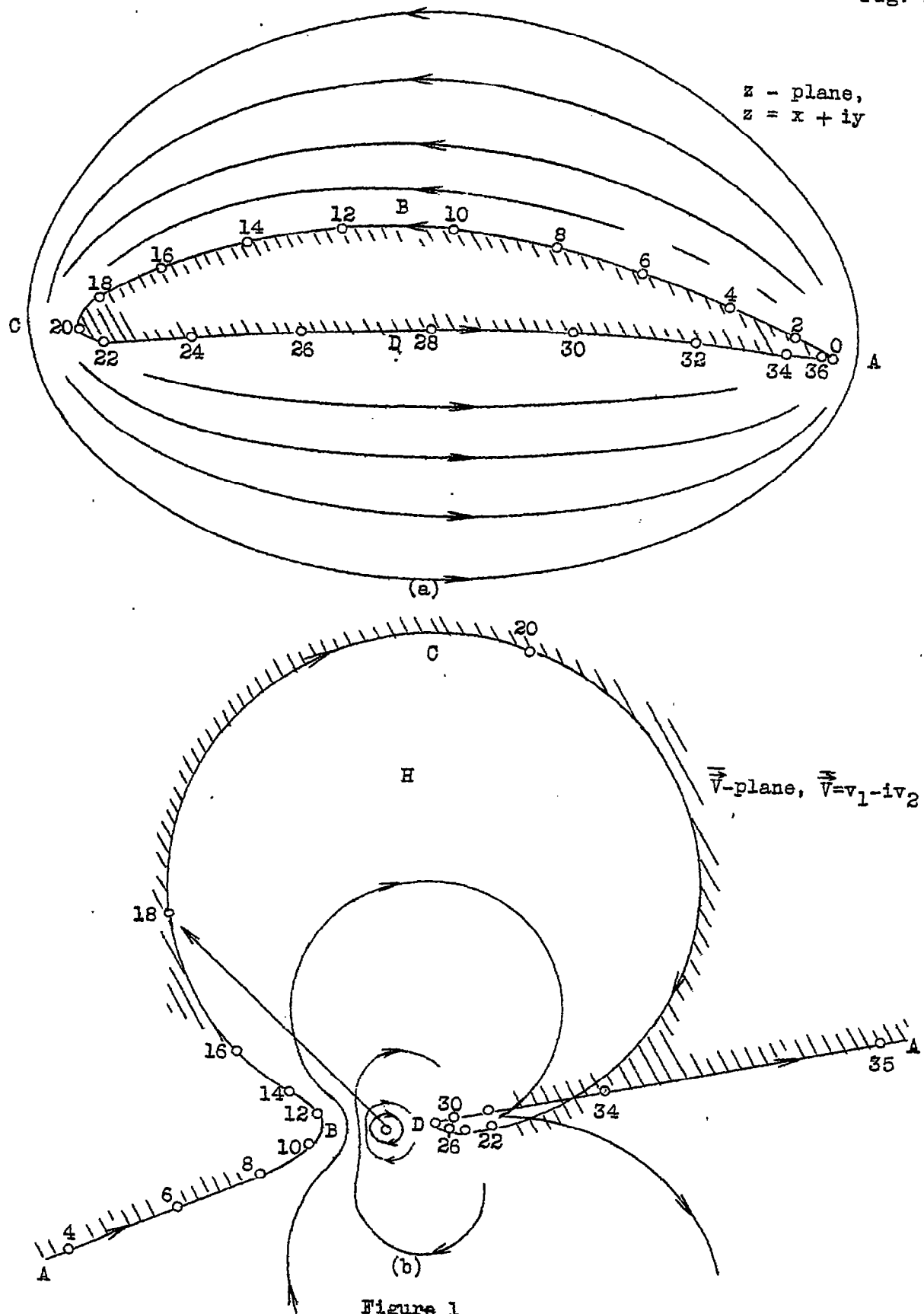


Table IIa
The values of F , H , $Q^{(1)}$ for $k = -0.5$

2λ	M	T	v/a_0	$H(2\lambda)$	$F(2\lambda)$	$Q^{(1)}(2\lambda)$
-	0	1	0	1	0	0
-2.82	0.265	0.964	0.272	1.000	0.0007	-0.0299
-1.96	0.403	0.915	0.430	1.001	0.0048	-0.0602
-1.612	0.473	0.881	0.518	1.002	0.0105	-0.0909
-1.51	0.497	0.867	0.551	1.003	0.0138	-0.1115
-1.388	0.528	0.849	0.593	1.004	0.0187	-0.1267
-0.79	0.690	0.723	0.861	1.015	0.1135	-0.3009
-0.394	0.820	0.572	1.165	1.046	0.6430	-0.7033
-0.188	0.901	0.433	1.440	1.111	3.6330	-1.5871
-0.132	0.921	0.388	1.527	1.145	6.9763	-2.2682

Table IIb. The values of F , H , $Q^{(n)}$, $R^{(n)}$ for $k = 1.4$

2λ	M	v/a_0	F	H	$Q^{(1)}$	$Q^{(2)}$	$Q^{(3)}$	$Q^{(4)}$	$R^{(0)}$	$R^{(1)}$	$R^{(2)}$	$R^{(3)}$
$-\infty$.0000	.0000	.0000	1.0000	.0000	.0000	.0000	.0000		.0000	.0000	.0000
-3.8772	.1000	.0999	.0001	1.0000	.0000	.0001	-.0001	.0002	9.9600	.0010	-.0020	.0000
-2.5096	.2000	.1992	.0011	1.0002	-.0009	.0015	-.0013	.0008	4.9197	.0030	-.0103	.0148
-1.7327	.3000	.2972	.0064	1.0014	-.0056	.0085	-.0083	.0064	3.2141	.0090	-.0411	.0670
-1.2071	.4000	.3938	.0256	1.0042	-.0199	.0342	-.0395	.0374	2.3371	.0235	-.1203	.2307
-.8238	.5000	.4879	.0866	1.0110	-.0574	.1160	-.1680	.2116	1.7947	.0520	-.3152	.7600
-.6706	.5500	.5341	.1565	1.0167	-.0935	.2101	-.3496	.5211	1.5901	.0755	-.5082	1.4060
-.5364	.6000	.5795	.2839	1.0247	-.1501	.3823	-.7494	1.3527	1.4150	.1088	-.8277	2.6949
-.4204	.6500	.6242	.5245	1.0359	-.2401	.7089	-1.6847	3.7802	1.2620	.1566	-1.3796	5.4353
-.3203	.7000	.6680	1.0060	1.0515	-.3870	1.3663	-4.0770	11.7558	1.1254	.2275	-2.3946	11.8168
-.2207	.7500	.7110	2.0623	1.0811	-.6959	2.8304	-12.7121	51.4676				
-.1615	.8000	.7532	4.6583	1.1049	-1.0896	6.4088	-35.1357	194.4562	.8847	.5248	-9.0769	81.6373
-.1015	.8500	.7945	12.5662	1.1517	-2.0186	17.4338	-149.6938	954.643	.7717	.8743	-21.9464	306.6745
-.0535	.9000	.8349	46.6378	1.2275	-4.3787	65.3783	-997.3960	16412.652	.6555	1.6804	-71.6427	1772.3621
.0000	1.0000	.9129	∞	∞	$-\infty$	∞	$-\infty$	∞	.0000	∞	$-\infty$	∞



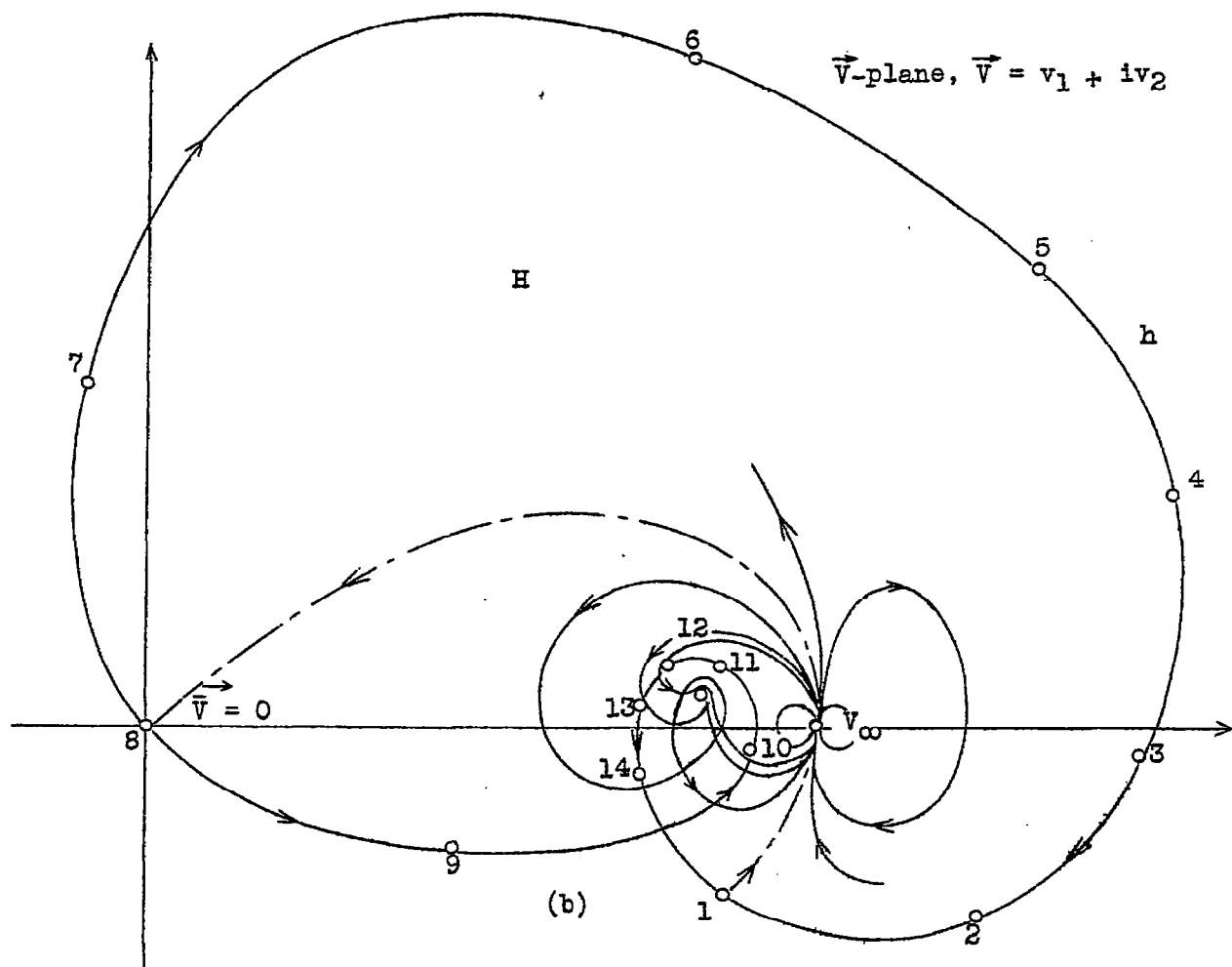
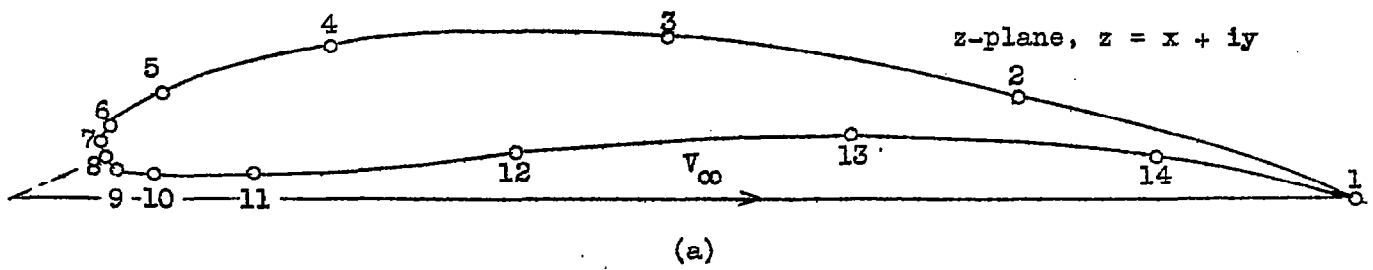


Figure 2

Z-plane

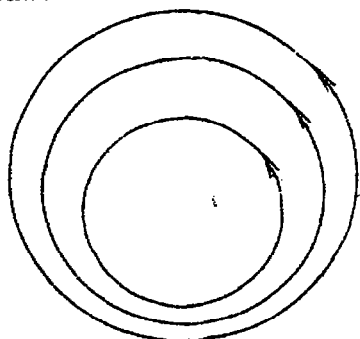


Figure 3

Z-plane

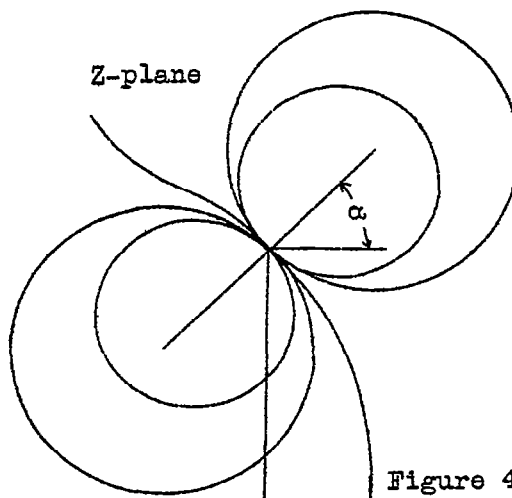


Figure 4

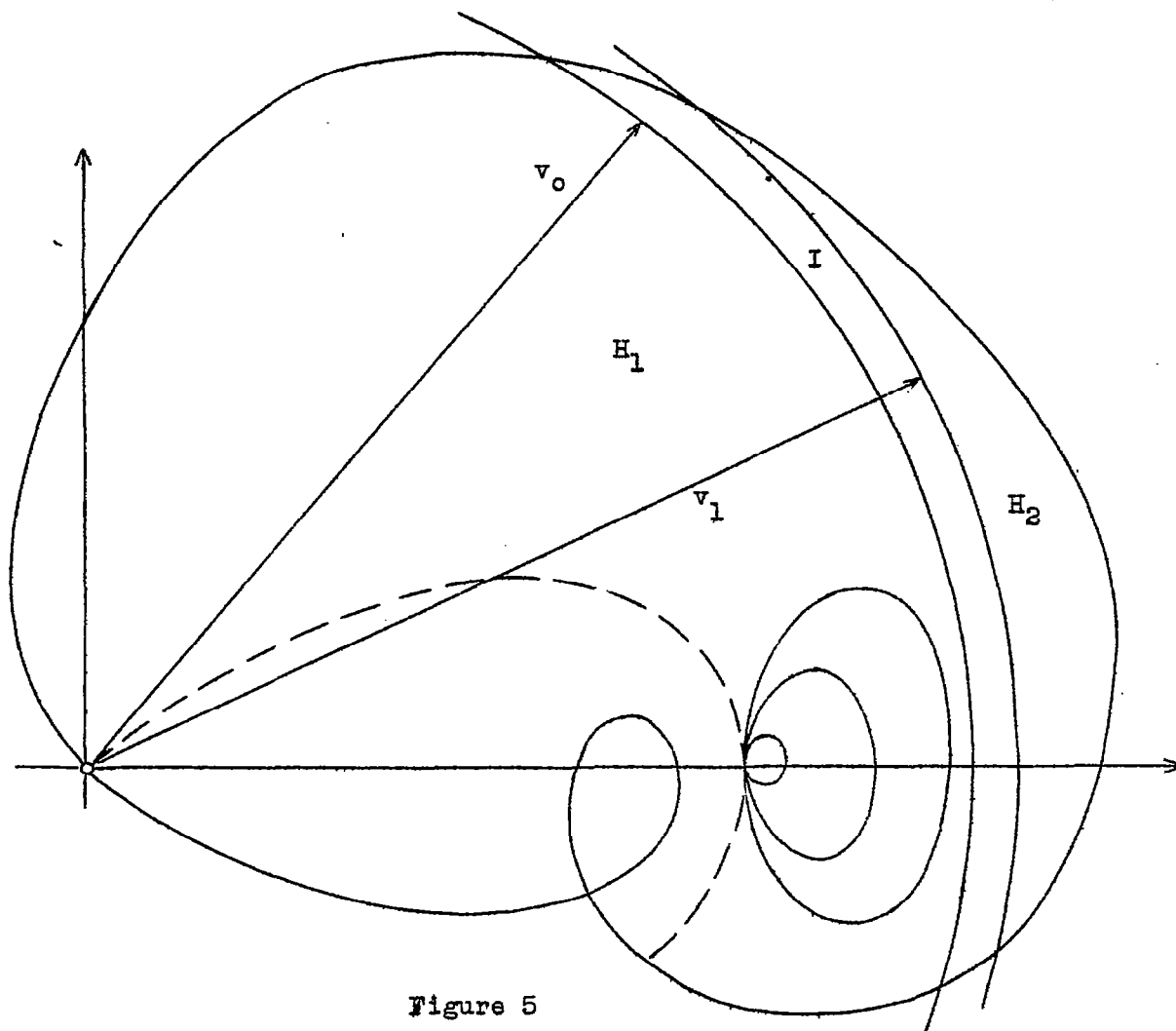


Figure 5